University of Anbar
College of Engineering
Dept. of Electrical Engineering

Control Theory I
Assist. Prof. Dr. Yousif Al Mashhadany
2019 - 2020

Lecture No. Four

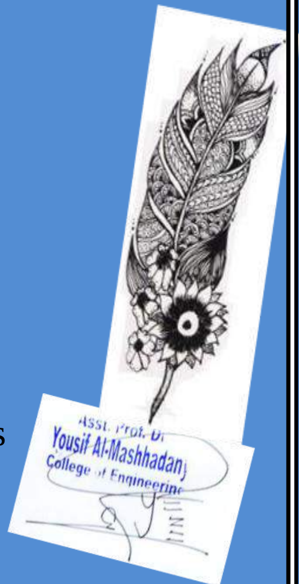
Signal Flow Graphs

This lecture will discusses the following topics

1. Introduction.
2. Flow-Graph Definitions.
3. Rules of Signal flow graph:
4. Signal flow graph for control system.
5. State Transition Signal Flow Graph

4.6. Simplification for the system of dual inputs

4.7. Matlab program for signal flow graph



4.1. Introduction:

The block diagram is a useful tool for simplifying the representation of a system. The block diagram has only one feedback loop and may be categorized as simple block diagrams. When it has two, three, etc, feedback loops; thus it is not a simple system. When intercoupling exists between feedback loops, and when a system has more than one input and one output, the control system and block diagram are more complex. Having the block diagram simplifies the analysis of complex system. Such an analysis can be even further simplified by using a signal flow graph (SFG), which looks like a simplified block diagram.

An SFG is a diagram that represents a set of simultaneous equations .It consists of a graph in which nodes are connected by directed branches. The nodes represent each of the system variables. A branch connected between two nodes acts as a one-way signal multiplier: the direction of signal flow is indicated by an arrow placed on the branch, and the multiplication of general matrix block diagram representing the state and output equations. factor (transmittance or transfer function) is indicated by a letter placed near the arrow. Thus, in Fig.4.1, the branch transmits the signal x_1 from left to right and multiplies it by the quantity a in the process. The quantity a is the transmittance, or transfer function. It may also be indicated by $a=t_{12}$, where the subscripts show that the signal flow is fromnode1to node 2.

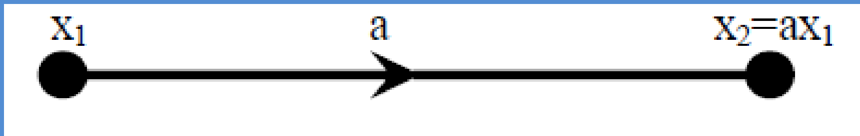


Fig.4.1. Signal flow graph for x2=ax1

4.2. Flow Graph Definitions.

The analysis of flow graph is achieved by mathematic process of anode that it performs two functions:

1. Addition of the signals on all incoming branches

2. Transmission of the total node signal (the sum of all incoming signals) to all outgoing branches these functions are illustrated in the graph of Fig.4.2, which represents the equations

$w = au + bv, \quad x = cw, y = dw$ (4.1) Three types of nodes are of

particular interest:

Source nodes (independent nodes).These represent independent variables and have only outgoing branches. In Fig.4.2, nodes u and v are source nodes.

Sink nodes (dependent nodes). These represent dependent variables and have only incoming branches. In Fig.4.2, nodes x and y are sink nodes.

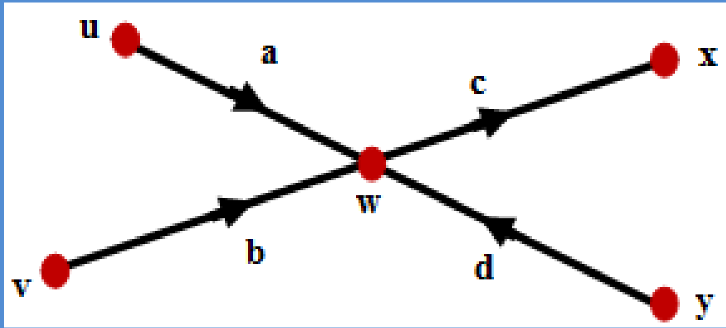


Fig.4.2. Signal flow graph for equation(4.1)

Mixed nodes (general nodes). These have both incoming and outgoing branches. In Fig.4.2, node w is a mixed node. A mixed node may be treated as a sink node by adding an outgoing branch of unity transmittance, as shown in Fig.4.3, for the equation

$$x = au + bv, \ \& \ w = cx = cau + cbv \qquad (4.2)$$

A path is any connected sequence of branches whose arrows are in the same direction. A forward path between two nodes is one that follows the arrows of successive branches and in which a node appears only once. In Fig.4.2, the path uwx is a forward path between the nodes u and x.

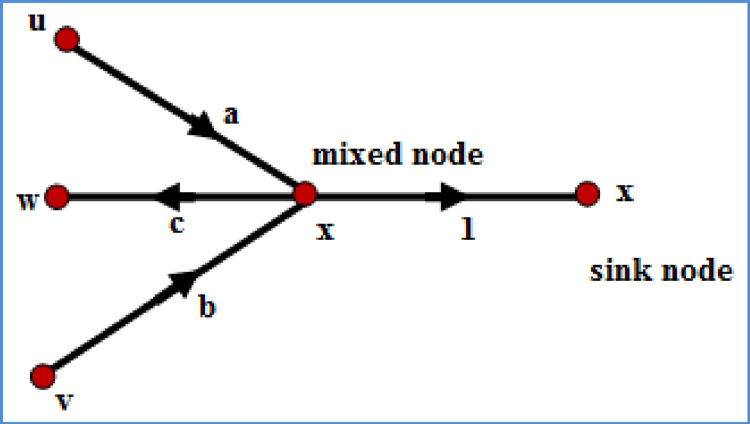


Fig.4.3. Mixed and sink nodes for a variable

Input Node (Source): An input node is a node that has only outgoing branches.

Output Node (Sink): An output node is a node that has only incoming branches. However, this condition is not always readily met by an output node.

Path: A path is any collection of a continuous succession of branches traversed in the same direction. The definition of a path is entirely general, since it does not prevent any node from being traversed more than once.

Forward Path: A forward path is a path that starts at an input node and ends at an output node and along which no node is traversed more than once.

Loop: A loop is a path that originates and terminates on the same node and along which no other node is encountered more than once.

Forward-Path Gain: The forward-path gain is the path gain of a forward path.

Loop Gain: The loop gain is the path gain of a loop.

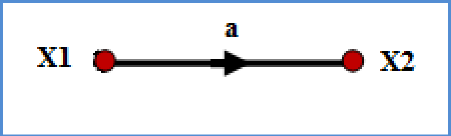
Nontouching Loops: Two parts of an SFG are nontouching if they do not share a common node.

.3. Rules of Signal flow graph

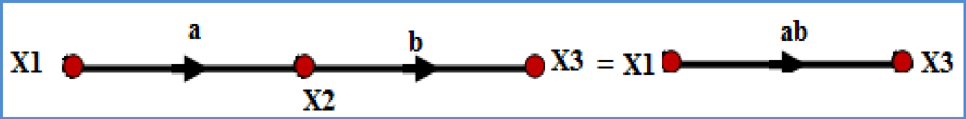
When constructing an SFG, junction points, or nodes, are used to represent variables. The nodes are connected by line segments called branches, according to the cause-and-effect equations. The branches have associated branch gains and directions. A signal can transmit through a branch only in the direction of the arrow.

1. The value of a node with one incoming branch, as shown below is

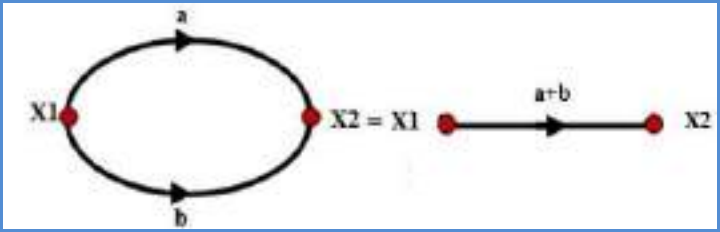
$$X_2 = aX_1 \quad (4.3)$$



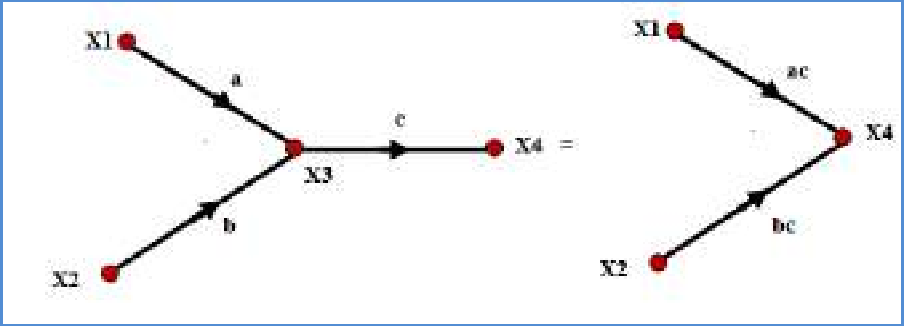
2. The total transmittance of cascaded branches is equal to the product of all branch transmittances. Cascaded branches can be combined into a single branch by multiplying the transmittances ,as shown below.



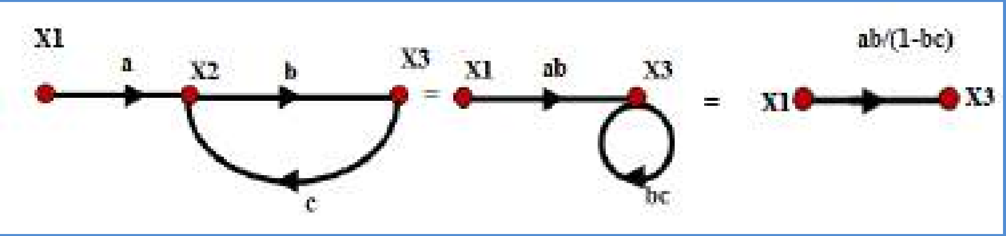
3. parallel branches may be combined by the transmittances, as shown below.



4. A mixed mode may be eliminated as shown below



5. A loop may be eliminated as shown below



The derivation of last relationship can be explained as in the following equations.

$$\left.\begin{aligned} X3 &= bX2 \\ X2 &= aX1 + cX3 \\ X3 = abX1 + bcX3 &\Rightarrow X3 - bcX3 = abX1 \\ X3(1-bc) = abX1 &\Rightarrow X3 = \frac{ab}{1-bc}X1 \end{aligned}\right\} \quad (4.4)$$

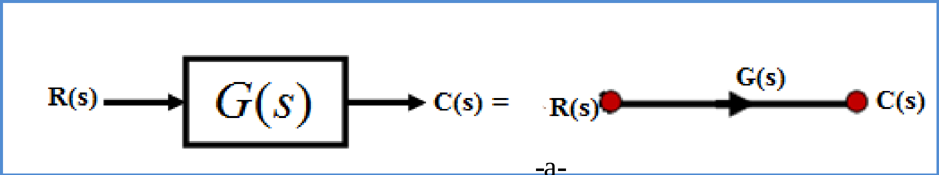
6.Signal flow graph (SFG) applies only to linear systems.

7.The equations for which an SFG is drawn must be algebraic equations in the form of cause and effect.

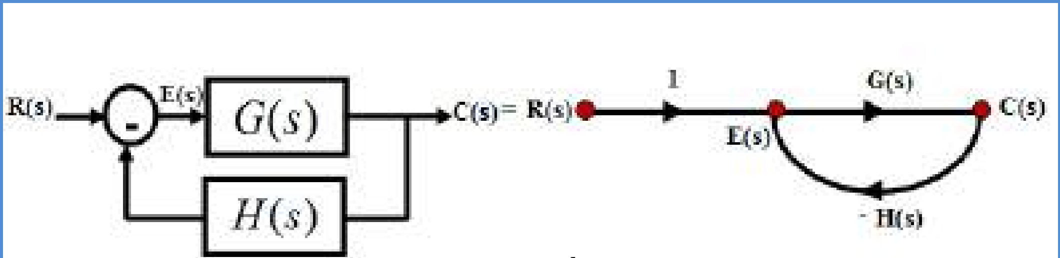
8. Nodes are used to represent variables. Normally, the nodes are arranged from left to right, from the input to the output, following a succession of cause-and-effect relations through the system.

4.4. Signal flow graph for control system

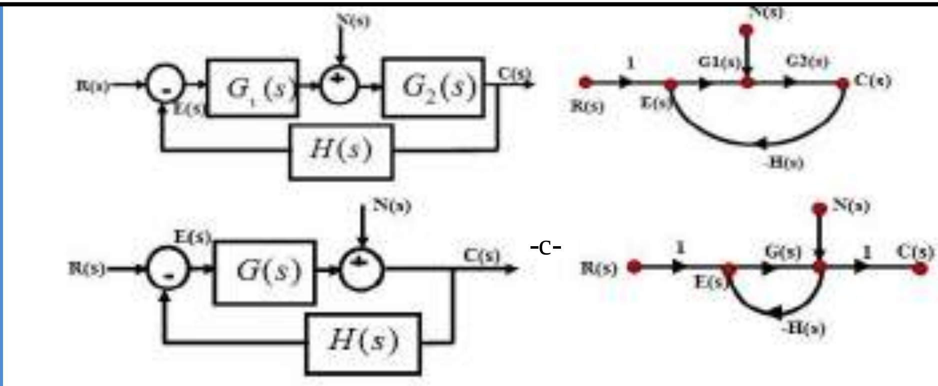
Some signal flow graphs of simple control system are shown in Fig.4.4. For such simple graphs, the closed loop transfer function C(s)/R(s) can be obtained easily by inspection. For more complicated signal flow graphs, Mason’s gain formula is quite useful.



-a-

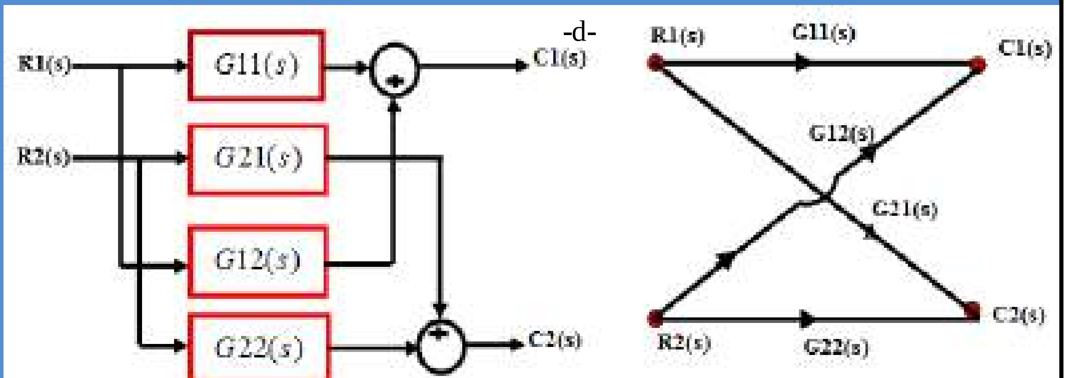


-b-


N(s)
R(s)
E(s)
$G_1(s)$
$G_2(s)$
C(s)
H(s)
1
G1(s)
G2(s)
-H(s)
G(s)

-c-

-d-


R1(s)
R2(s)
G11(s)
G21(s)
G12(s)
G22(s)
C1(s)
C2(s)

-e-

Fig.4.4.(a-e). Signal flow graph forms for simple control system.

-e-

In many practical cases e wish to determine the relationship between an input variable and an output variable of the signal flow graph. The transmittance between an input node and an output node is the overall gain, or overall transmittance, between these two nodes. Mason's gain formula, which is applicable to the overall gain, is given by

$$P = \frac{1}{\Delta} \sum_{p} P_k \Delta_k \qquad (4.5)$$

Where

P_k= path gain or transmittance of the Kth forward path

Δ_k= cofactor of the *Kth* forward path determinant for the graph with the loops touching the *Kth* forward path removed.

Δ= determinant of the graph.

Δ=1-(sum of all different loop gains)+(sum of gain products of all possible combinations of two nontouching loops)- (sum of gain products of all possible combinations of three nontouching loops)+…….

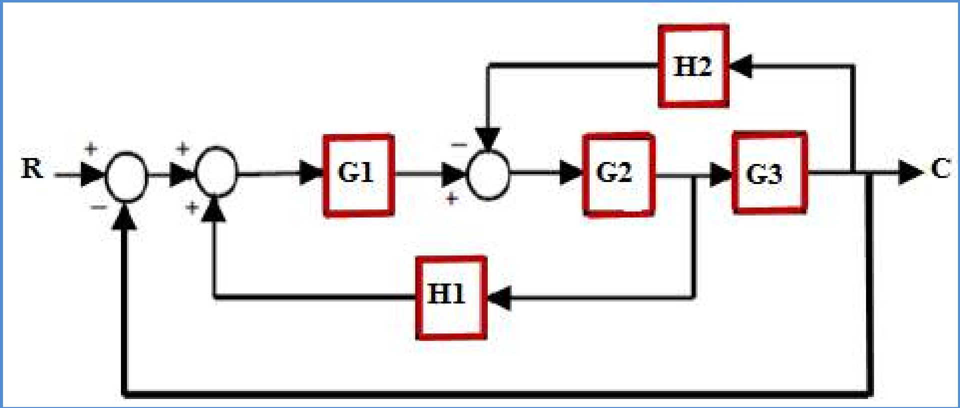
$$\Delta = 1 - \sum_{a} L_a + \sum_{b,c} L_b L_c - \sum_{d,e,f} L_d L_e L_f + ...$$

$\sum_{a} L_a =$ sum of all different loop gains

$\sum_{b,c} L_b L_c =$ sum of gain products of all possible combinations of two nonteaching loops

$\sum_{d,e,f} L_d L_e L_f =$ sum of gain products of all possible combinations of three nonteaching loops.

Example(1): Find the transfer function C(s)/R(s) for the system block diagram shown below by using Mason' gain formula?



Solution:

In the system there is only one forward path between the input R(s) and the output C(s). The forward path gain is,

$$p_1 = G_1 G_2 G_3$$

From the signal flow graph, we see that there are three individual loops.

The gains of these loops are;

$$L_1 = G_1 G_2 H_1$$

$$L_2 = -G_3 G_2 H_2$$

$$L_3 = -G_1 G_2 G_3$$ `

Note that since all three loops have a common branch, there are no non-touching loops; hence, the determinant is Δ given by

$$\Delta = 1 - (L_1 + L_2 + L_3) = 1 - G_1 G_2 H_1 + G_3 G_2 H_2 + G_1 G_2 G_3$$

The factor Δ_1 of the determinant along the forward path connecting the input node and output node is obtained by removing the loops that touch this path.

Since path P_1 touch all loops, we have ; $\Delta_1=1$

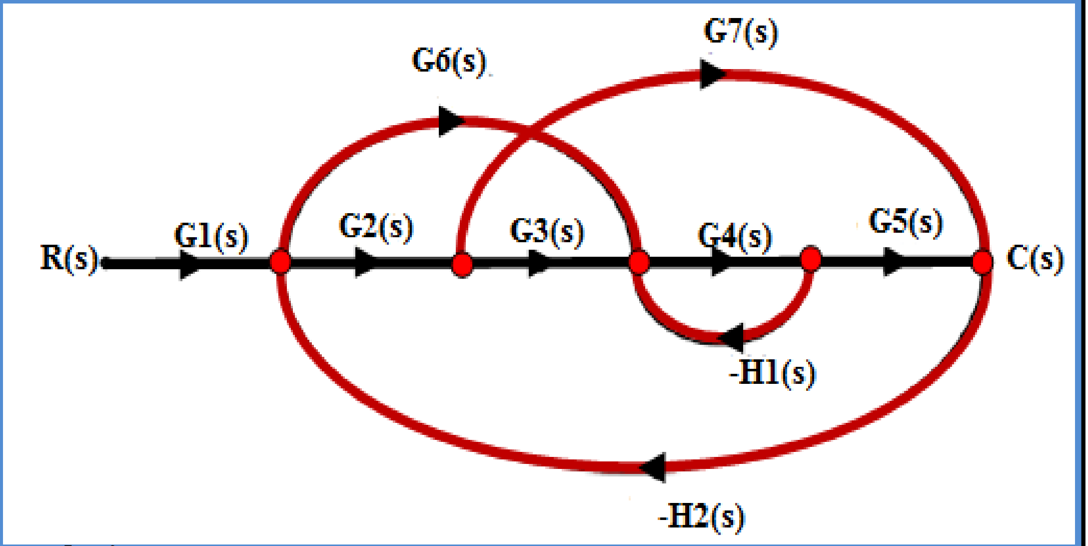
Therefore the overall transfer function of the closed loop system is given by:

$$\frac{C(s)}{R(s)} = \frac{P_1\Delta_1}{\Delta}$$

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1-G_1G_2H_1+G_2G_3H_2+G_1G_2G_3}$$

HW. Try to obtain the transfer function by block diagram reduction and compare the results.

Example (2): Consider the SFG of a system shown in the following figure. Obtain the closed-loop transfer function C(s)/R(s) by use of Mason's gain formula?



Solution:

In this system there are three forward paths between the input and the output.

$$p_1 = G_1G_2G_3G_4G_5$$

$$p_2 = G_1G_6G_4G_5$$

$$p_3 = G_1G_6G_7$$

There are four individual loops in this system

$$L_1 = -G_4H_1$$

$$L_2 = -G_2G_7H_2$$

$$L_3 = -G_6G_4G_5H_2$$

$$L_4 = -G_2G_3G_4G_5H_2$$

Loop L_1 does not touch Loop L_2 and there are no nontouching loops in this system just L_1 and L_2 so that the determinant of the system Δ will be

$$\Delta=1-(L_1 +L_2+ L_3+ L_4)+ L_1 L_2$$

The factor Δ_1 is obtain from Δ by removing the loops that touch p_1 ,therefore by removing L_1,L_2 , L_3, L_4 and $L_1 L_2$,the factor $\Delta_1 =1$.

Similarly by eliminating all loops in Δ that touch p_2.

$$\Delta_2 =1$$

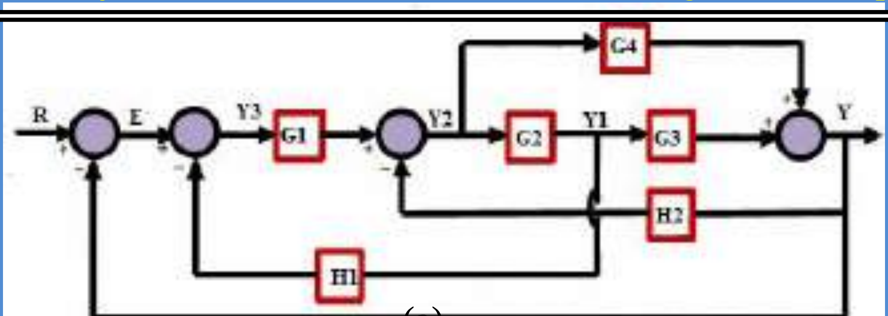
Δ_3 can be obtained by removing ,L_2 , L_3, L_4 and $L_1 L_2$ from the Δ that touch p_3

$\Delta_3=1- L_1$; The transfer function of the closed loop system C(s)/R(s)

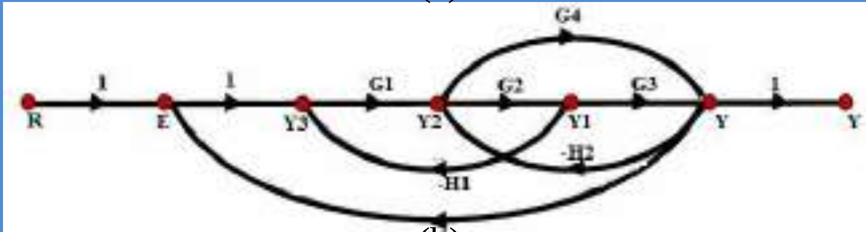
$$\frac{C(s)}{R(s)} = \frac{1}{\Delta}\left(p_1\Delta_1 + p_2\Delta_2 + p_3\Delta_3\right)$$

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4G_5 + G_1G_6G_4G_5 + G_1G_2G_7(1+G_4H_1)}{1+G_4H_1 + G_2G_7H_2 + G_6G_4G_5H_2 + G_4H_1G_2G_7H_2 + G_2G_3G_4G_5H_2}$$

To illustrate how an equivalent SFG of a block diagram is constructed and how the gain formula is applied to a block diagram, consider the block diagram shown in Fig.4.5.(a). The equivalent SFG of the system is shown in Fig. 4.5.(b).



(a)



(b)

Fig. 4. 5. (a) Block diagram of a control system, (b) Equivalent signal flow graph.

Notice that since a node on the SFG is interpreted as the summing point of all incoming signals to the node, the negative feedbacks on the block diagram are represented by assigning negative gains to the feedback paths on the SFG. First we can identify the forward paths and loops in the system and their corresponding gains. That is: forward path gain

$$p_1 = G_1G_2G_3$$

$$p_2 = G_1G_4$$

Loop gains

$$L_1 = -G_1G_2H_1$$

$$L_2 = -G_2G_3H_2$$

$$L_3 = -G_1 G_2 G_3$$

$$L_4 = -G_4 H_2$$

$$L_5 = -G_1 G_4$$

The closed loop transfer function of the system is obtained by applying Mason' gain formula to the SFG or using the block diagram reduction.

$$\frac{Y(s)}{R(s)} = \frac{G_1 G_2 G_3 + G_1 G_4}{\Delta}$$

$$\Delta = 1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 + G_4 H_2 + G_1 G_4$$

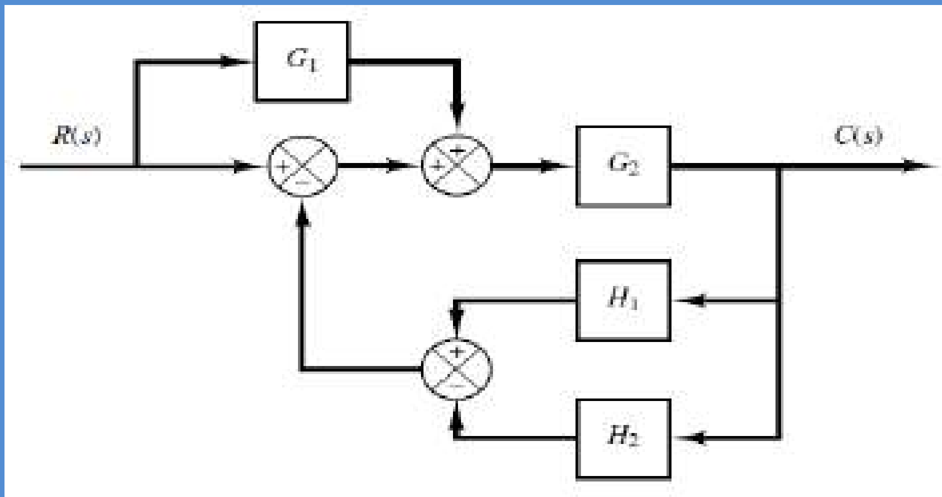
$$\frac{Y(s)}{R(s)} = \frac{G_1 G_2 G_3 + G_1 G_4}{1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 + G_4 H_2 + G_1 G_4}$$

Quiz No. Two:

Q1. A. Obtain a state-space equation and transfer function for the system defined by:

$$\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = 2\dddot{u} + \ddot{u} + \dot{u} + 2u$$

Q1. B. Simplify the block diagram shown in Figure below and obtain the closed-loop transfer function C(s)/R(s)?



4.5. State Transition Signal Flow Graph:

The state transition SFG or, more simply, the state diagram, is a simulation diagram for a system of equations and includes the initial conditions of the states. Since the state diagram in the Laplace domain satisfies the rules of Mason's SFG, it can be used to obtain the transfer function of the system and the transition equation. The basic elements used in a simulation diagram are a gain, a summer, and an integrator. The signal-flow representation in the Laplace domain for an integrator is obtained as follows:

$$\dot{x}_1(t) = x_2(t); \Rightarrow sx_1(t) = x_2(t) \qquad (4.6)$$

$$X_1(t) = \frac{x_2(t)}{s} + \frac{x_1(t_0)}{s} \qquad (4.7)$$

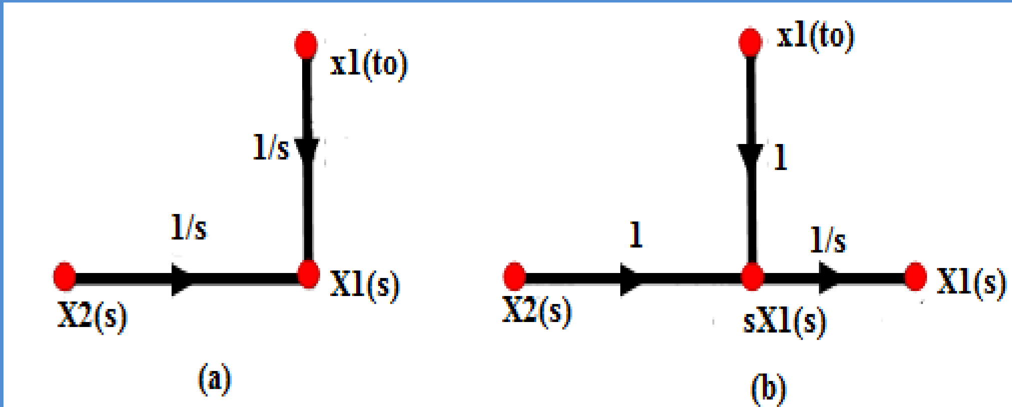
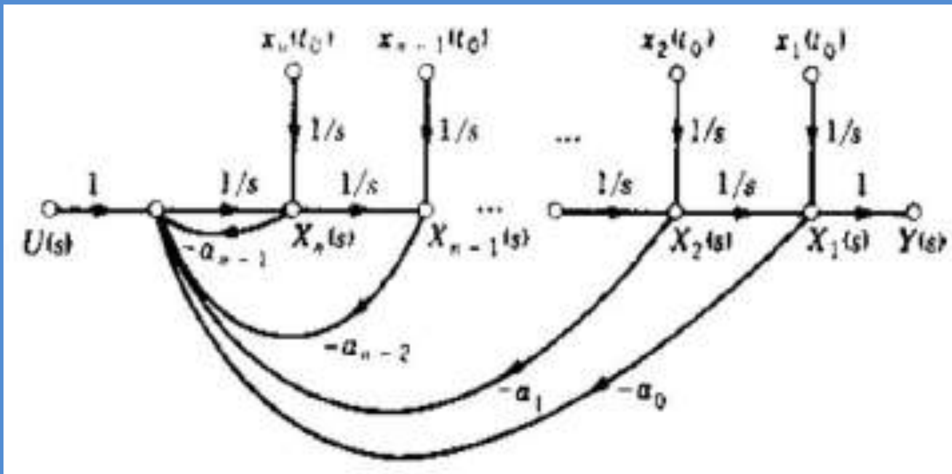


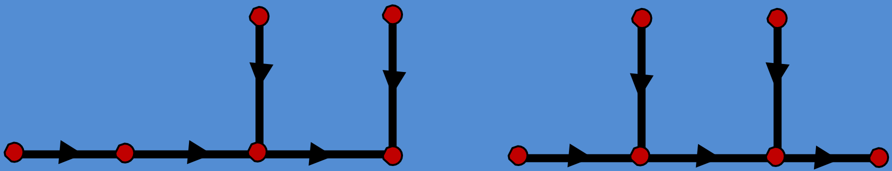
Fig.4.6. Representations of an integrator in the Laplace domain in a signal flow graph

The above equation may be represented either by Fig.4.6.a or Fig.4.6.b. A differential equation that contains no derivatives of the input, as given by:

$$D^n y + a_{n-1} D^{n-1} y + a_1 D \; y + a_0 y = u \qquad (4.8)$$



Eample4: For the following equation find:



$$\ddot{y} + \frac{R}{L}\dot{y} + \frac{1}{LC}y = \frac{1}{LC}u$$

(a) Draw the state diagram. (b) Determine the state transition equation?

Solution:

(a) The state Diagram, Fig. below, includes two integrators, because this is a second-order equation. The state variables are selected as the phase variables that are the outputs of the integrators, that is, x_1=y and x_2= x_1.

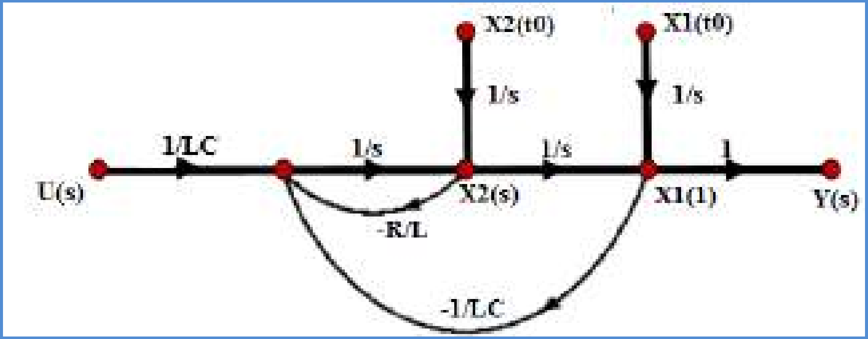
(b) The state transition equations are obtained by applying the Mason gain formula with the three inputs u, $x_1(t_0)$, and $x_2(t_0)$:

$$X_1(s) = \frac{s^{-1}(1 + s^{-1}R / L}{\Delta(s)} x_1(t_0) + \frac{s^{-2}}{\Delta(s)} x_2(t_0) + \frac{s^{-2} / LC}{\Delta(s)} U(s)$$

$$X_2(s) = \frac{s^{-2} / LC}{\Delta(s)} x_1(t_0) + \frac{s^{-1}}{\Delta(s)} x_2(t_0) + \frac{s^{-1} / LC}{\Delta(s)} U(s)$$

$$\Delta(s) = 1 + \frac{s^{-1}R}{L} + \frac{s^{-2}}{LC}$$

The signal flow graph for this system is



After simplification these equations become

$$X(s) = \begin{bmatrix} X1(s) \\ X2(s) \end{bmatrix}$$

$$X(s) = \frac{1}{s^2 + (R/L)s + 1/LC} \left\{ \begin{bmatrix} s + R/L & 1 \\ -1/LC & s \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} + \begin{bmatrix} 1/LC \\ s/LC \end{bmatrix} U(s) \right\}$$

4.6. Simplification for the system of dual inputs:

According to the principle of superposition theory, we must find the output by considering one input at a time and cancelled another courses. For the system is shown in Fig.4.7, we find C1(s)/R(s), and then $C_2(s)/D_1(s)$, and $C_3(s)/D_2(s)$, the final output of system is achieved by summation of these three inputs; $C = C_1 + C_2 + C_3$

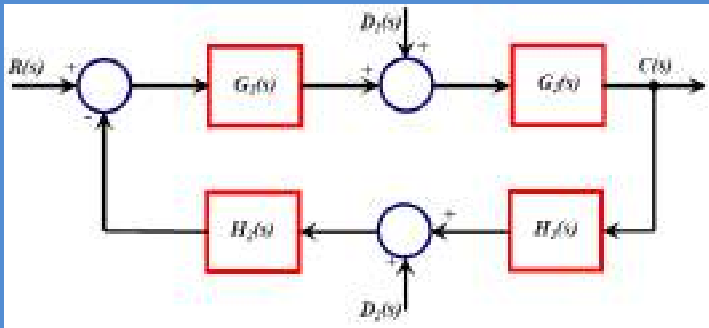
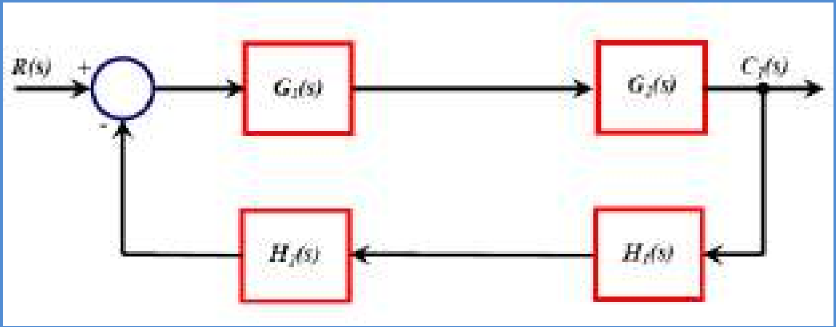


Fig.4.7. Block diagram for dual inputs control system.

Output due to input R(s):

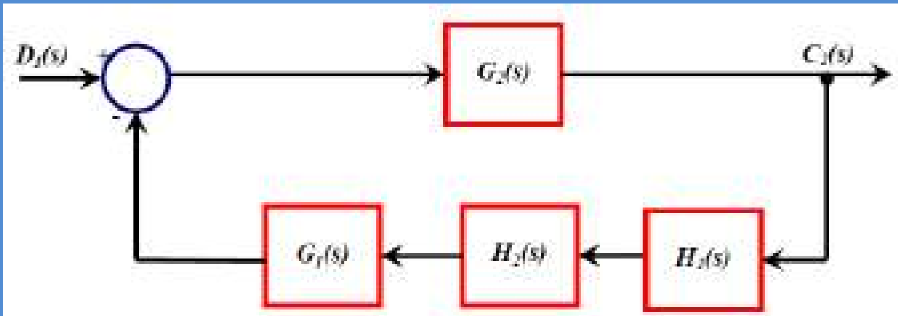
Let D1(s)=0 and D2(s)=0, Hence the system becomes; the transfer function for this block is ;



$$\frac{C_1(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H_1(s)H_2(s)} \qquad (4.9)$$

Output due to input D1(s):

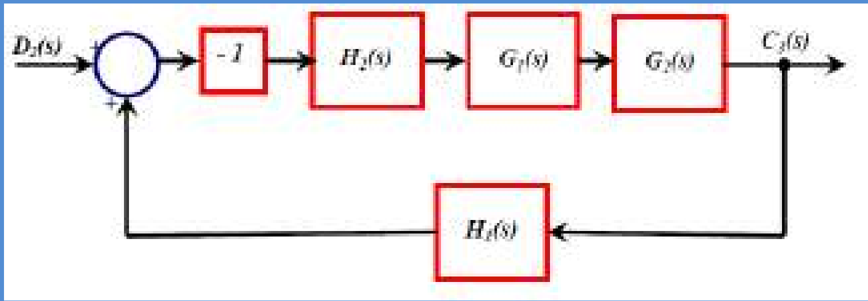
Let R(s)=0 and D2(s)=0, Hence the system becomes; the transfer function for this block is ;



$$\frac{C_2(s)}{D_1(s)} = \frac{G_2(s)}{1 + G_1(s)H_1(s)H_2(s)} \qquad (4.10)$$

Output due to input D2 (s):

Let R1(s)=0 and D1(s)=0, Hence the system becomes; the transfer function for this block is ;



$$\frac{C_3(s)}{D_2(s)} = \frac{-G_2(s)G_1(s)H_2(s)}{1+G_1(s)G_2H_1(s)H_2(s)} \qquad (4.11)$$

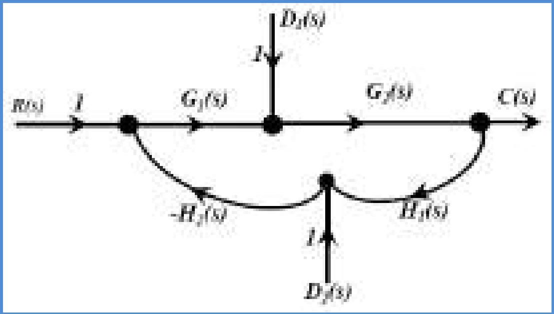
The total output of the system is the summation of outputs with respect to a corresponding input:

$C = C_1 + C_2 + C_3$

$$C = \frac{RG_1(s)G_2(s)}{1+G_1(s)G_2(s)H_1(s)H_2(s)} + \frac{D_1G_2(s)}{1+G_1(s)H_1(s)H_2(s)} + \frac{-D_2G_2(s)G_1(s)H_2(s)}{1+G_1(s)G_2H_1(s)H_2(s)}$$

$$C = \frac{G_2(s)[RG_1(s)}{1+G_1(s)G_2(s)H_1(s)H_2(s)} + \frac{D_1}{1+G_1(s)H_1(s)H_2(s)} + \frac{-D_2G_1(s)H_2(s)]}{1+G_1(s)G_2H_1(s)H_2(s)}$$

The solution by using signal flow graph: Firstly draw the signal flow graph diagram for original B.D;

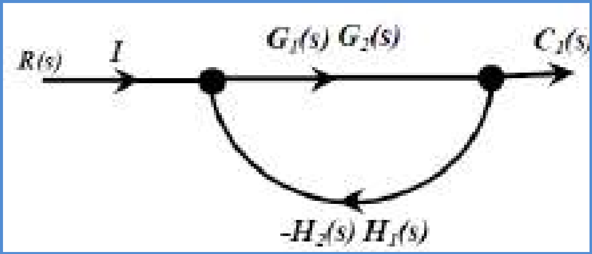


$$P = \frac{1}{\Delta}\sum_{p} P_k\Delta_k$$

When apply the superposition theory can be get;

Output due to input R(s):

Let D1(s)=0 and D2(s)=0, the signal flow graph becomes;



By apply the Mason's formula**:**

$$p_1 = G_1(s)G_2(s)$$

$$L_1 = -G_1(s)G_2(s)H_1(s)H_2(s)$$

$$L_2 = 0$$

$$\Delta = 1 - L_1 = 1 + G_1(s)G_2(s)H_1(s)H_2(s)$$

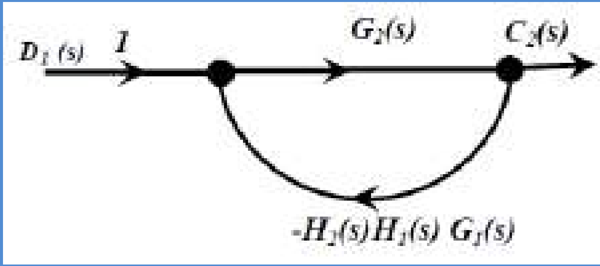
$$\Delta_1 = 1$$

$$P_R = \frac{1}{\Delta}(P_1\Delta_1)$$

$$P_R = \frac{C_1(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)}$$

Output due to input D1(s):

Let R(s)=0 and D2(s)=0, the signal flow graph becomes;



$$p_1 = G_2(s)$$

$$L_1 = -G_1(s)G_2(s)H_1(s)H_2(s)$$

$$\Delta = 1 - L_1 = 1 + G_1(s)G_2(s)H_1(s)H_2(s)$$

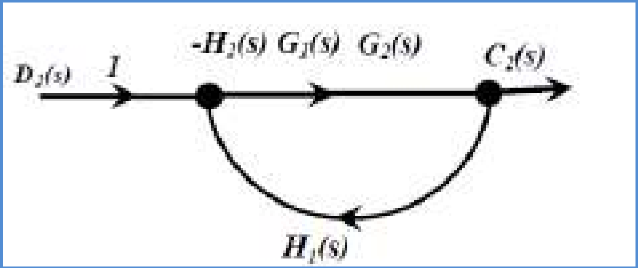
$$\Delta_1 = 1$$

$$P_{D_1} = \frac{1}{\Delta}(P_1\Delta_1)$$

$$P_R = \frac{C_2(s)}{D_1(s)} = \frac{G_2(s)}{1+G_1(s)G_2(s)H_1(s)H_2(s)}$$

Output due to input D2(s):

Let R(s)=0 and D1(s)=0, the signal flow graph becomes;



$$p_1 = -G_1(s)G_2(s)H_2(s)$$

$$L_1 = -G_1(s)G_2(s)H_1(s)H_2(s)$$

$$\Delta = 1 - L_1 = 1 + G_1(s)G_2(s)H_1(s)H_2(s)$$

$$\Delta_1 = 1$$

$$P_{D2} = \frac{1}{\Delta}(P_1\Delta_1)$$

$$P_{D2} = \frac{C_2(s)}{D_2(s)} = \frac{-G_1(s)G_2(s)H_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)}$$

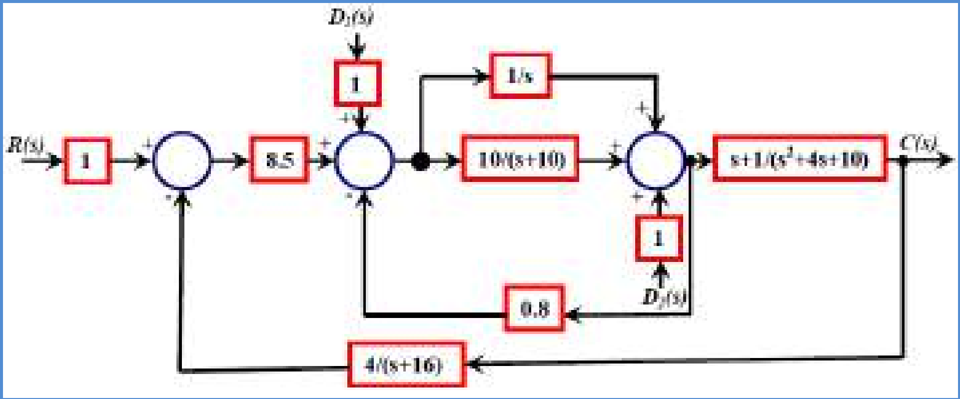
$$P_{total} = P_R + P_{D1} + P_{D2}$$

$$P_{total} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)} R + \frac{G_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)} D_1 + \frac{-G_1(s)G_2(s)H_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)} D_2$$

$$C=\frac{G_2(s)[G_1(s)R+D_1--G_1(s)H_2(s)D_2]}{1+G_1(s)G_2(s)H_1(s)H_2(s)}$$

Note: Where the denominator represents the polynomial of the system therefore; it has the same form in all inputs affect and in all form where the polynomial refer to clc's of the system (i.e. A matrix in state space form).

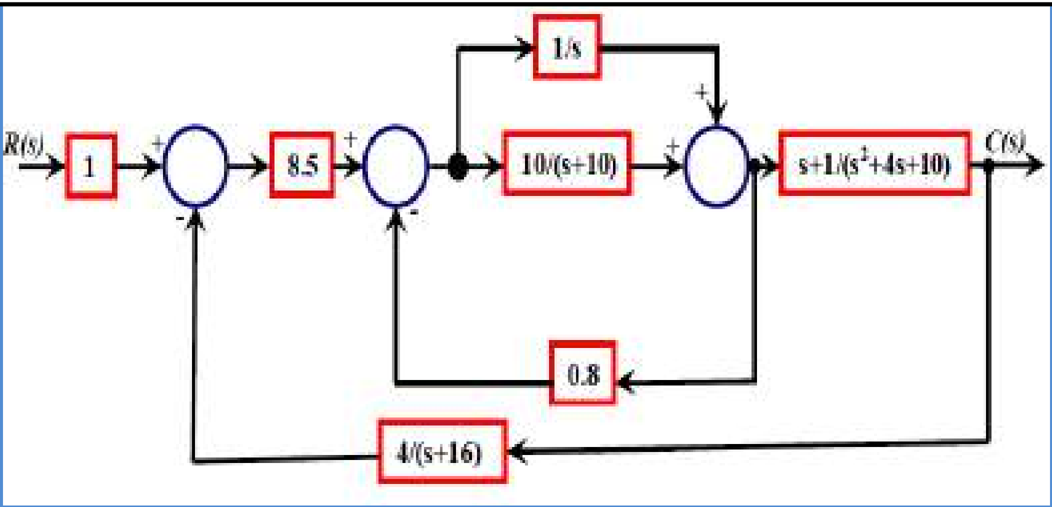
Example5: Find the T.F for the block diagram shown below?



Solution:

By using superposition theory:

1. C/R(s) by set D1 and D2=0; the block becomes:



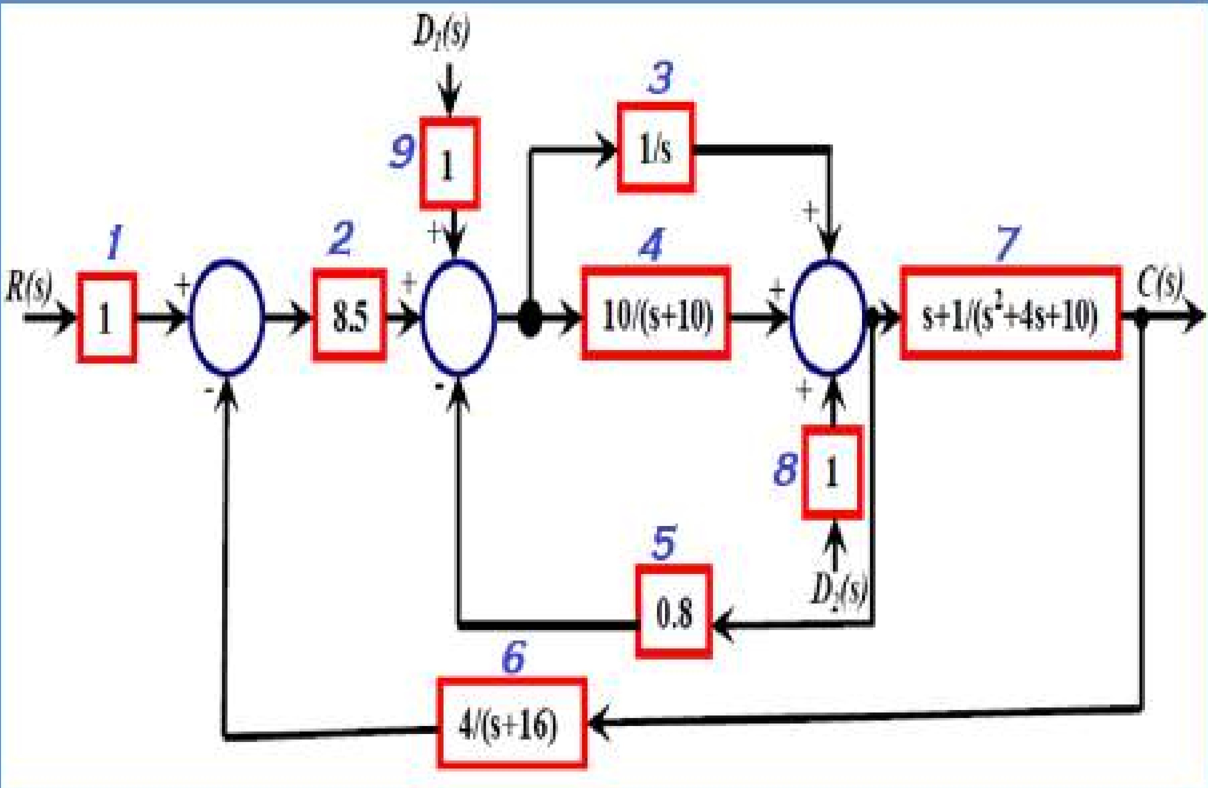
And can be continue with solution by using B.D.R and By using S.F.G. and this is a home work. From all the above examples, we can see that all loops and forward paths are touching in this case.

As a general rule, if there are no nontouching loops and forward paths in the block diagram or SFG of the system, then the Mason' gain formula can be putted in a simplified formula, as shown next.

$$\frac{C}{R} = \sum \frac{forward\ \ path\, gains}{1 - loop\ gains}$$

4.7. Matlab program for signal flow graph

Also all the complex solution can be minimize by using MATLAB with the following program, for the control system that is shown in Example 5, we can find the solution by using Matlab program. The following program for MISO system to get the final transfer



D1(s)
9
1
3
1/s
1
2
4
7
R(s)
1
8.5
10/((s+10)
s+1/(s2+4s+10)
C(s)
8
1
5
0.8
D2(s)
6
4/(s+16)

function which can be get analytically by block diagram reduction or signal flow graph.

Program in MATLAB to find T.F. of MISO system:

```
n1=[1];d1=[1];
n2=[8.5];d2=[1];
n3=[1];d3=[1 0];
n4=[10];d4=[1 10];
n5=[.8];d5=[1];
n6=[4];d6=[1 16];
```

```
n7=[1 1];d7=[1 4 10];
n8=[1];d8=[1];
n9=[1];d9=[1];
nblocks=9;
blkbuild;
q=[ 1 0 0 0
2 1 -6 0
3 2 -5 8
```

```
4 2 -5 8
5 3 4 9
6 7 0 0
7 3 4 9
8 0 0 0
9 0 0 0 ];
iu=9;
iy=7;
```

```
[ac,bc,cc,dc]=connect(a,b,c,d,q,iu,iy);
[num,den]=ss2tf(ac,bc,cc,dc,1)
printsys (num,den)
%i/p R & O/P C
% 4.2633e-014 s^4 + 93.5 s^3 + 1674.5 s^2 + 2941 s +
1360
%T.F. R/C= ------------------------------------------------------
% s^5 + 38.8 s^4 + 458 s^3 + 2085.2 s^2 + 4314 s + 1620
%i/p D1 & O/P C
% 2.8422e-014 s^4 + 11 s^3 + 197 s^2 + 346 s + 160
```

```
% T.F C/D1= -----------------------------------------------------
% s^5 + 38.8 s^4 + 458 s^3 + 2085.2 s^2 + 4314 s + 1620
%i/p D1 & O/P C
% s^4 + 27 s^3 + 186 s^2 + 160 s - 1.728e-011
% T.F. D2/C -------------------------------------------------------
% s^5 + 38.8 s^4 + 458 s^3 + 2085.2 s^2 + 4314 s + 1620
```

University of Anbar
College of Engineering
Dept. of Electrical Engineering

Control Theory I
Assist. Prof. Dr. Yousif Al Mashhadany
2019 - 2020

Lecture No. Five

Transient Response Analysis

This lecture will discusses the following topics

5.1. **Analysis of typical control system.**

5.2. **Samples of systems Response.**

5.3. **Test inputs.**

5.4. **Second –order systems and T.R. specifications.**

5.5. **Parameters of transient-response.**

5.6. **Solved problem.**

5.1. Analysis of a typical control system

Consider a first differential equation : $\frac{dx(t)}{dt}+ax(t)=u(t)$

This may be the equation of a physical system with input u(t) and output x(t) i)function) is that part of t response which occurs near t =0 and then decays .this part of t response is due to the characteristics of the system only .

ii) Steady state part (s.s) (particular integral) is that part of the response which is present throughout the period t=0 to t= .but at t this the complete solution because the transient part is absent.

The nature of steady state response depends of external input only .complete solution=Tr part +S.S part

i) Auxiliary equation (characteristic equation):

$$m+a=0 \Rightarrow m=-a$$

Transient part Ae^{-at}

ii) Steady state part

let u(t)=U(constant) $\frac{dx(t)}{dt}=0$ at steady state:

$$aX_{ss}=U \Rightarrow X_{ss}=\frac{U}{a}$$

$$X(t) = Ae^{-at} + \frac{U}{a}$$

If we know x(t)=0 at t=0:

$$X(t) = \frac{U}{a}(1 - e^{-at})$$

At t=0 , x(t)=0

at t=∞ , x(t)=U/a

at t=1/a, x(t)=0.63 U/a

T=1/a =time constant

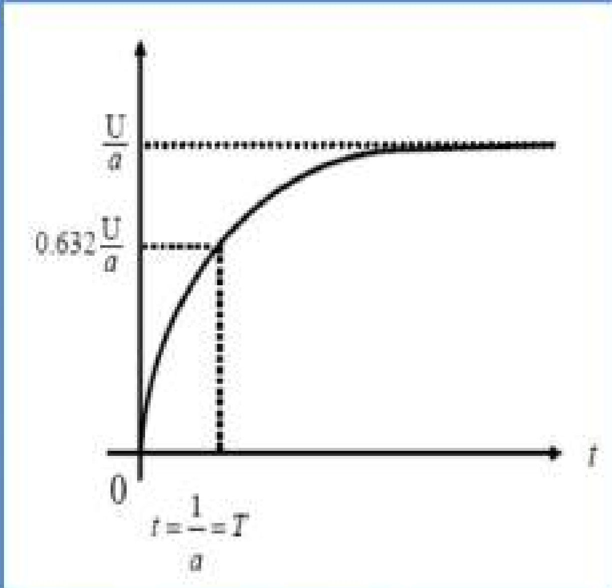


Fig. 5.1. Response of first order system

5.2. Samples of systems Response.

i) R-C , R-L circuit with constant voltage input.

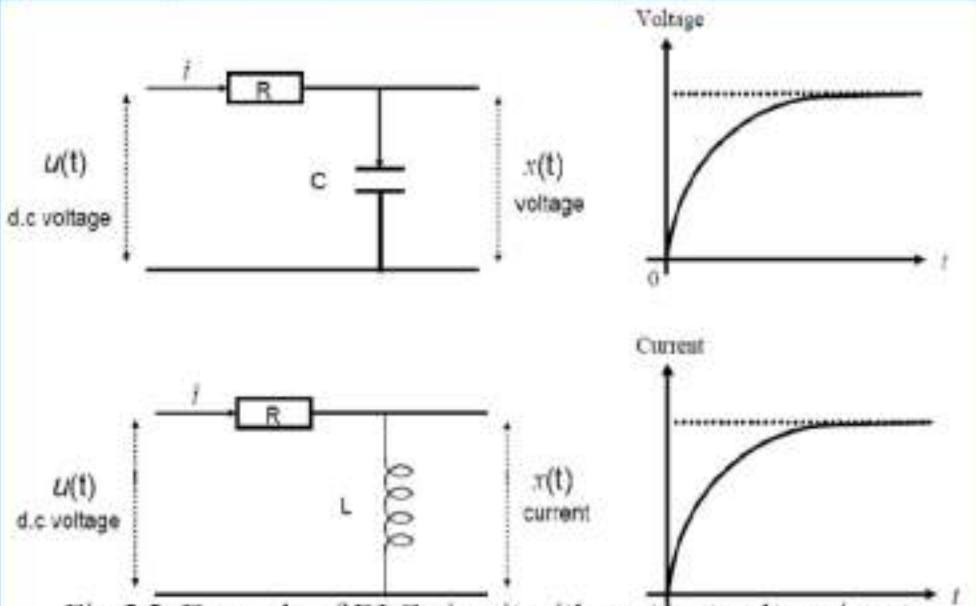
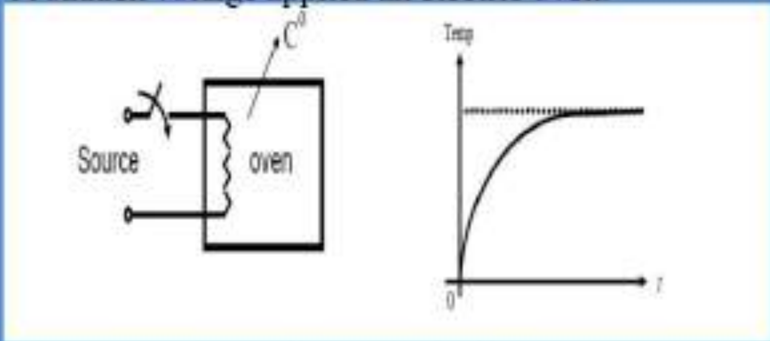
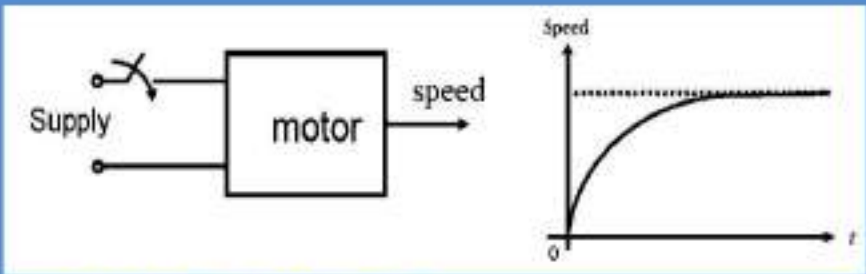


Fig.5.2. Example of RLC circuit with constant voltage input.

ii) A sudden voltage applied an electric oven.



iii) Constant supply voltage switched on to motor.



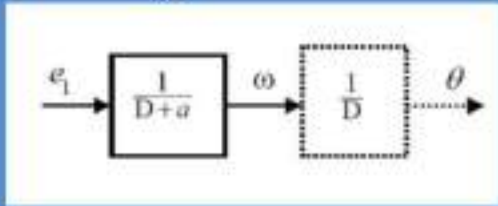
All these system may be represented by differential equation of first order.

SUCH SYSTEMS ARE CALLED FIRST ORDER SYSTEMS

Consider a D.C. motor operating with a constant field current i_f. If the input to the motor is taken as e_1 (armature voltage) and the output is taken as speed ω. The differential equation of the motor may be written as :- $\frac{d\omega(t)}{d(t)} + a\omega(t) = e_1(t)$

Using 'D' operator $D\omega(t) + a\omega(t) = e_1(t) : \frac{\omega(t)}{e_1(t)} = \frac{1}{D+a}$

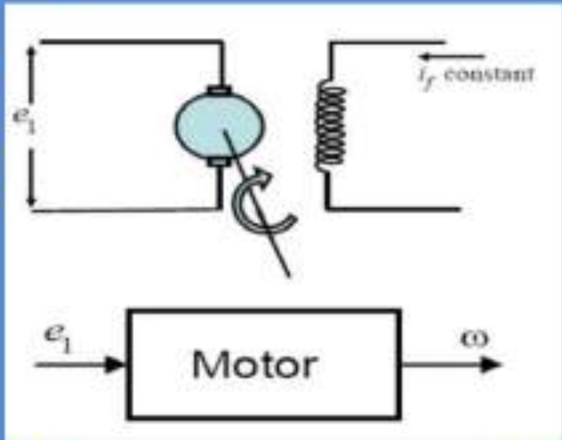
If we define θ as the output variable, $\omega = \frac{d\theta}{dt}$

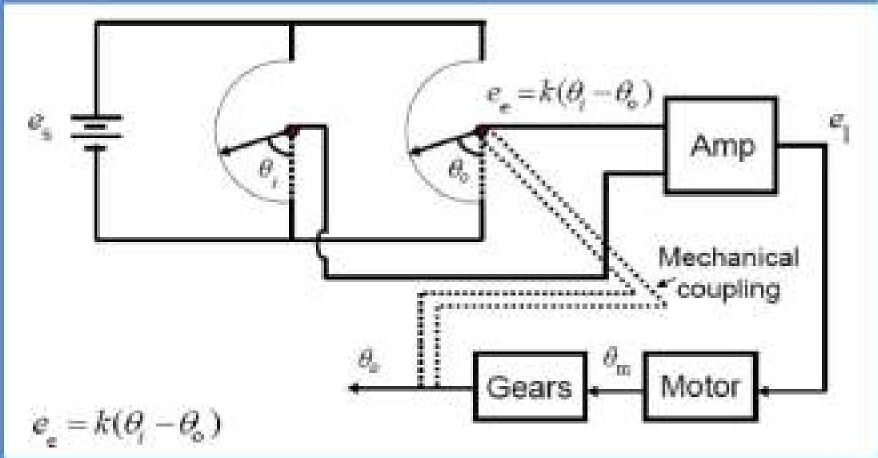


$$\frac{d^2\theta(t)}{dt^2} + a\frac{d\theta(t)}{dt} = e_1(t)$$

$$D^2\theta + aD\theta = e_1(t) \quad ;$$

$$\frac{\theta(t)}{e_1(t)} = \frac{1}{D(D+a)}$$

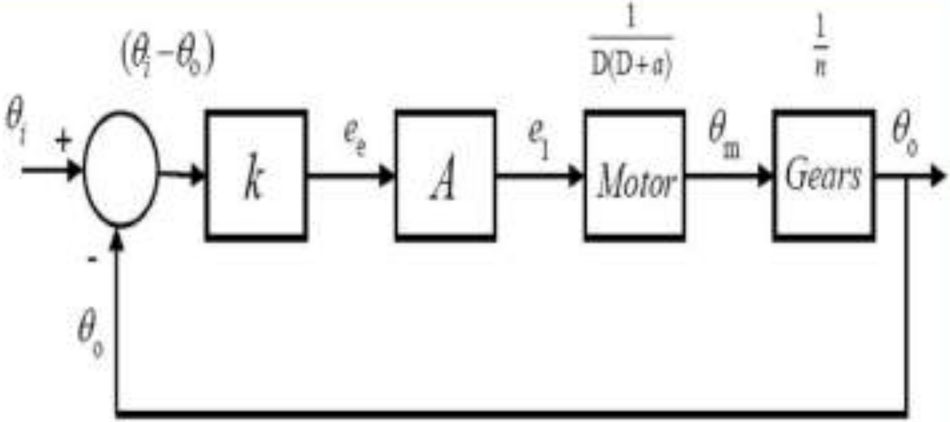




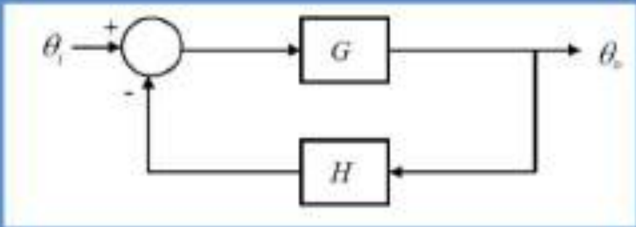
Analysis of a position control system:

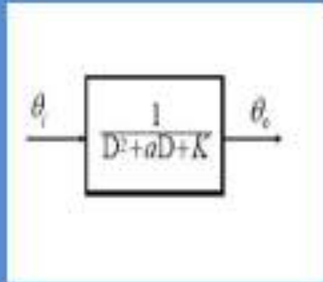
$$k = \frac{e_e}{\theta_i - \theta_o} \quad ; \quad k = \frac{e_s}{\theta_{max}}$$

Block diagram:



Simplified block diagram:





$$G = \left(k.A\frac{1}{D(D+a)}\frac{1}{n}\right) = \frac{K}{D(D+a)}$$

Where K = system gain

$$\frac{\theta_o}{\theta_i} = \frac{G}{1+GH} = \frac{\frac{K}{D(D+a)}}{1+\frac{K}{D(D+a)}} = \frac{K}{D^2+aD+K}$$

$$\frac{d^2\theta_o(t)}{dt^2} + a\frac{d\theta_o(t)}{dt} + K\theta_o(t) = k\theta_i(t)$$

a is parameter of the motor:

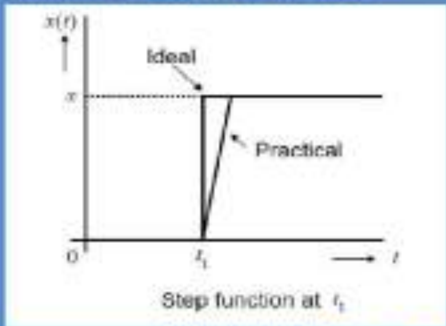
$$K = k.A\frac{1}{n}$$

Where A is amplifier gain and $\frac{1}{n}$ is gear ratio.

5.3. Test inputs

i) Step function: a step is a sudden change in the value of the physical quantity $x(t)$ from one level (usually zero) to another level, in zero time.

$$x(t) = \begin{cases} x & t > t_1 \\ 0 & t \leq t_1 \end{cases}$$

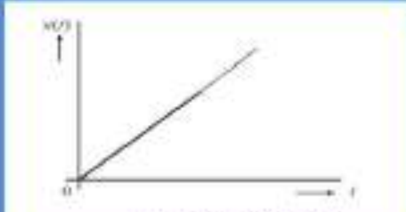


Unit step:

$$u(t) = \begin{cases} 1 & t > t_1 \\ 0 & t \le t_1 \end{cases}$$

ii) Ramp Function: ramp is a signal which starts from a zero level and increase linearly with espect to time.

$$x(t) = \begin{cases} kt & t > 0 \\ 0 & t \le 0 \end{cases}$$

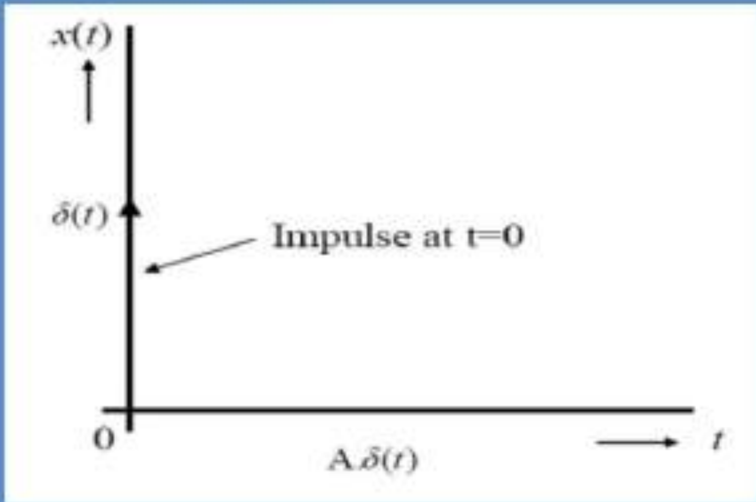


Ramp function at $t = 0$

iii) Pulse Function: a pulse may be considered as a step function which is present for limited period.

$$x(t) = \begin{cases} x & 0 < t \le T \\ 0 & elsewhere \end{cases}$$

iv) Impulse Function: if in the pulse, the width is decreased and the height is increased such that



$\lim_{\substack{T \to 0 \\ X \to \infty}} X.T = A$, the resulting function is impulse $A\delta(t)$.

5.4. Second –order systems and T.R. specifications.

Differential Equation of the C.L position control system:

$$\frac{d^2\theta_o(t)}{dt^2} + a\frac{d\theta_o(t)}{dt} + K\theta_o(t) = k\theta_i(t)$$

For step input , $\theta_i(t) = R \, , t > 0$

$$\frac{d^2\theta_o(t)}{dt^2} + a\frac{d\theta_o(t)}{dt} + K\theta_o(t) = kR$$

Solve the differential equation.

i) S.S solution $(\dot{\theta}_o(t) = \ddot{\theta}_o(t) = 0)$

$(\theta_o)_{s.s} = R$

ii) Transient solution

Auxiliary equation: $r^2 + ar + k = 0$

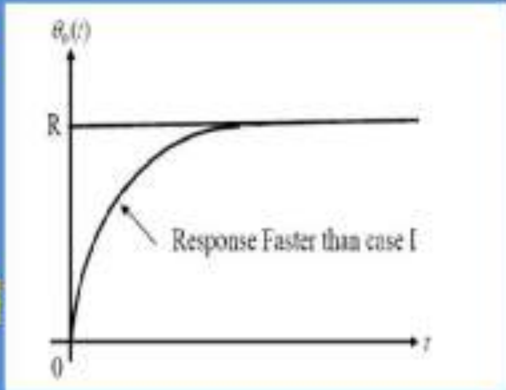
(characteristic equation) $r_1, r_2 = \frac{-a \pm \sqrt{a^2 - 4k}}{2}$

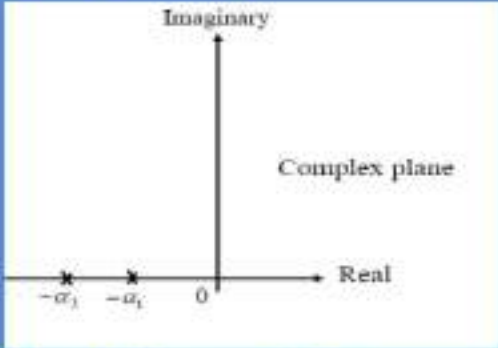
Case I: The two roots are distinct.

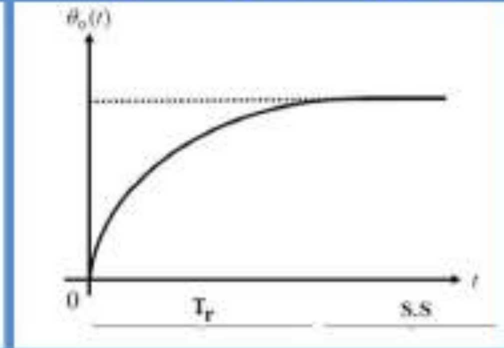
$$\boldsymbol{r_1, r_2 = -\alpha_1, -\alpha_2 \quad ; a^2 > 4k}$$

$$\boldsymbol{(\theta_o)_{Tr} = C_0 e^{-\alpha_1 t} + C_1 e^{-\alpha_2 t}}$$

$$\boldsymbol{\theta_o(t) = R + C_0 e^{-\alpha_1 t} + C_1 e^{-\alpha_2 t}}$$







Case II: Repeated Roots.

$$r_1, r_2 = -\alpha, -\alpha \quad ; \alpha^2 = 4k$$

$$(\theta_o)_{Tr} = (C_o + C_1 t)e^{-\alpha t}$$

$$\theta_o(t) = R + (C_o + C_1 t)e^{-\alpha t}$$

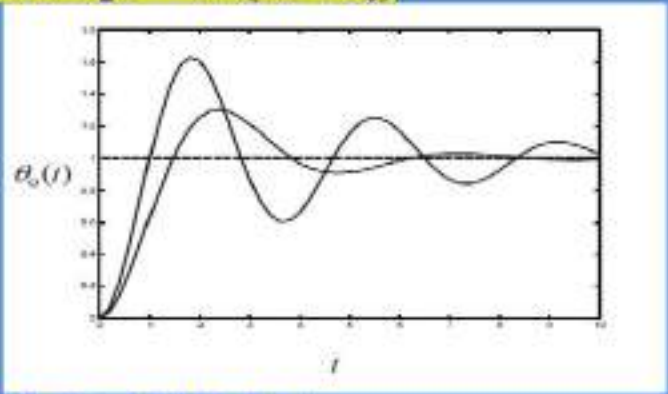
Case III: Complex Conjugate Roots.

$$r_1, r_2 = -\alpha \pm jw \quad ; \alpha^2 < 4k \; ;$$

$$(\theta_o)_{Tr} = e^{-\alpha t}(C_o \cos wt + C_1 \sin wt)$$

$$(\theta_o)_{Tr} = C_2 e^{-\alpha t} \sin(wt + C_3)$$

$$\theta_o(t) = R + C_2 e^{-\alpha t} \sin(wt + C_3)$$

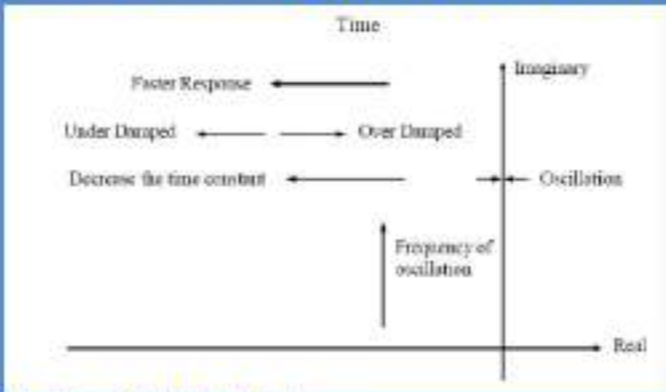


Response of Position Control System:

i) **Distinct Roots.** $\theta_o(t) = R + C_o e^{-\alpha_1 t} + C_1 e^{-\alpha_2 t}$

ii) **Repeated Roots.** $\theta_o(t) = R + (C_o + C_1 t)e^{-\alpha t}$

iii) **Complex Conjugate Roots.** $\theta_o(t) = R + C_2 e^{-\alpha t}\sin(wt + C_3)$;

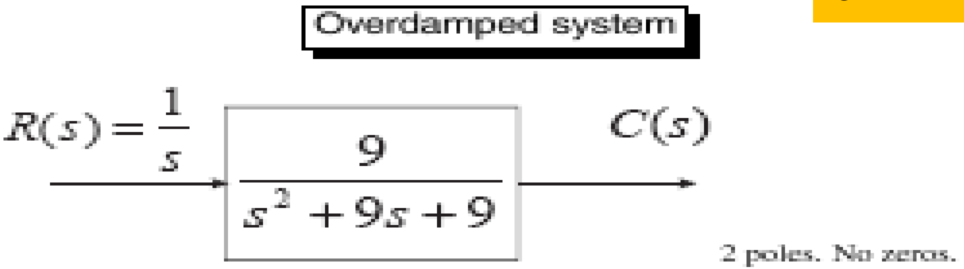


Overdamped Response

$$G(s) = \frac{b}{s^2 + as + b}$$

a = 9

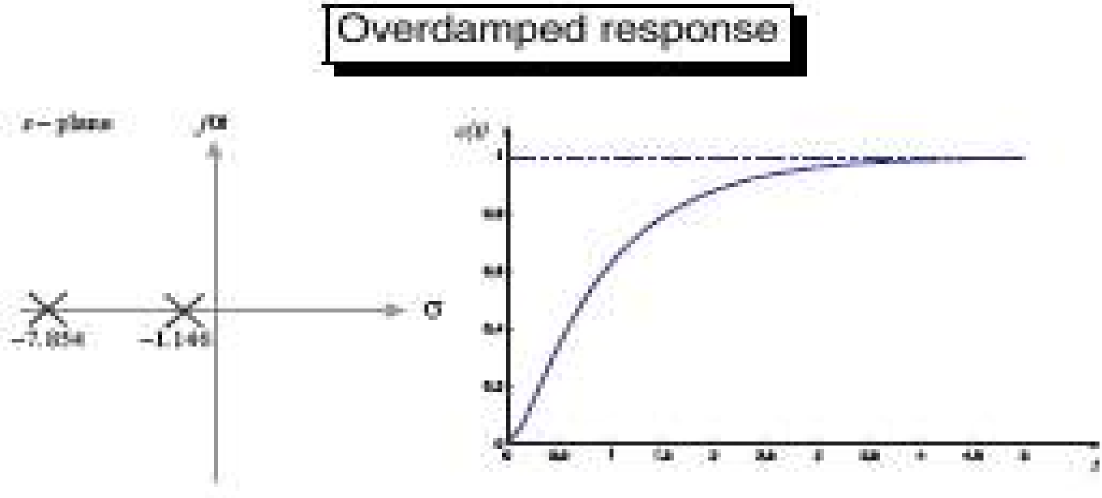
$\xi > 1$



$$C(s) = \frac{9}{s(s^2 + 9s + 9)} = \frac{9}{s(s + 7.854)(s + 1.146)}$$

s= 0; s = -7.854; s = -1.146 *(two real poles)*

$$c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$$



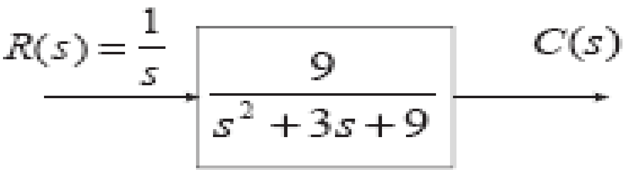
Underdamped Response

$$G(s) = \frac{b}{s^2 + as + b}$$

a = 3

$$0 < \xi < 1$$

Underdamped system

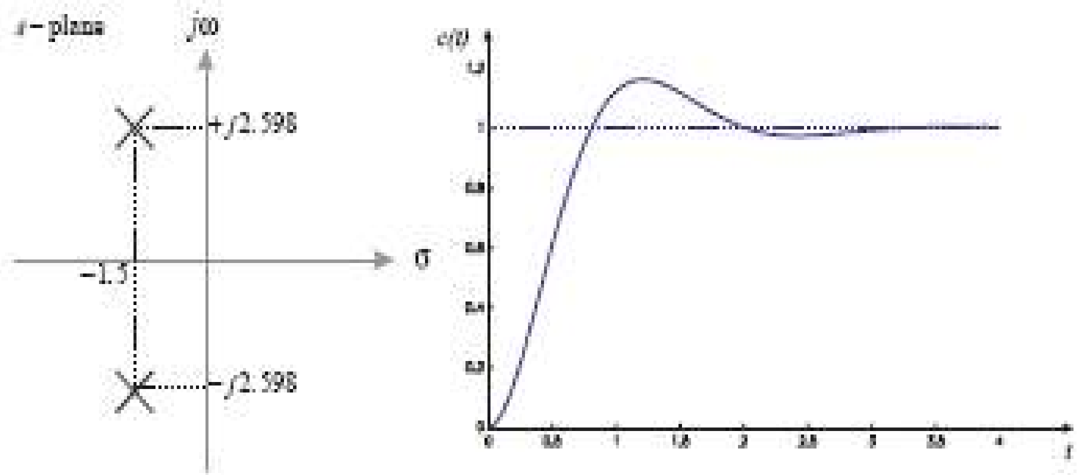


2 poles. No zeros.

$$c(t) = K_1 + e^{-1.5t}(K_2 \cos 2.598t + K_3 \sin 2.598t)$$

s = 0; s = -1.5 ± j2.598 *(two complex poles)*

Underdamped response

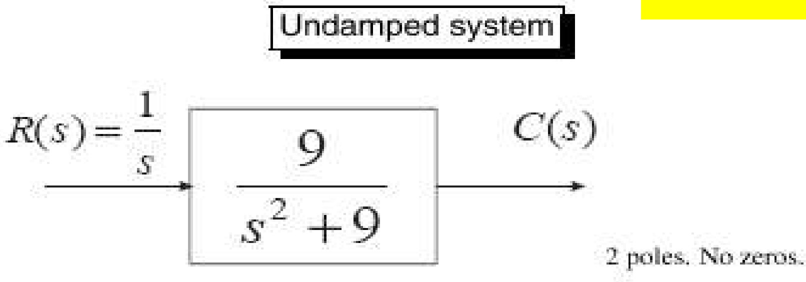


Undamped Response

$$G(s) = \frac{b}{s^2 + as + b}$$

$a = 0$

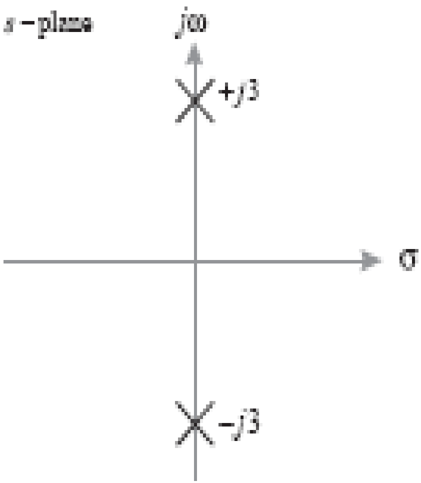
$\xi = 0$

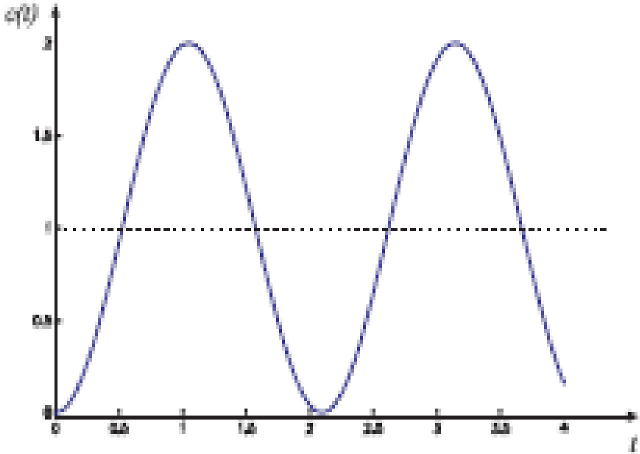


$$c(t) = K_1 + K_2 \cos 3t$$

s = 0; s = ± j3 *(two imaginary poles)*

Undamped response





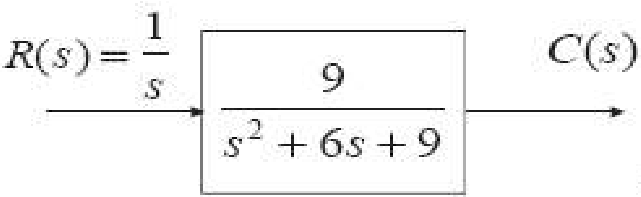
Critically Damped System

$$G(s) = \frac{b}{s^2 + as + b}$$

$a = 6$

$\xi = 1$

Critically Damped System

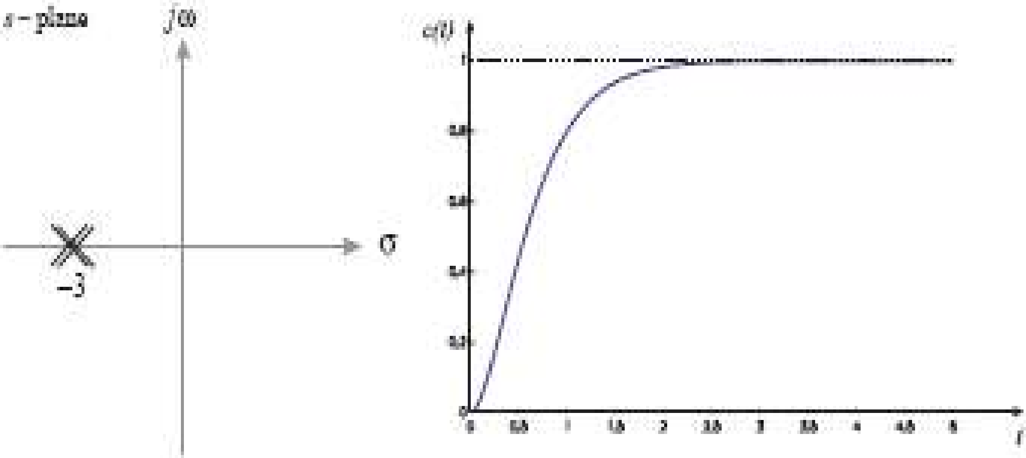


2 poles. No zeros.

$$c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}$$

S = 0; s = -3,-3 *(two real and equal poles)*

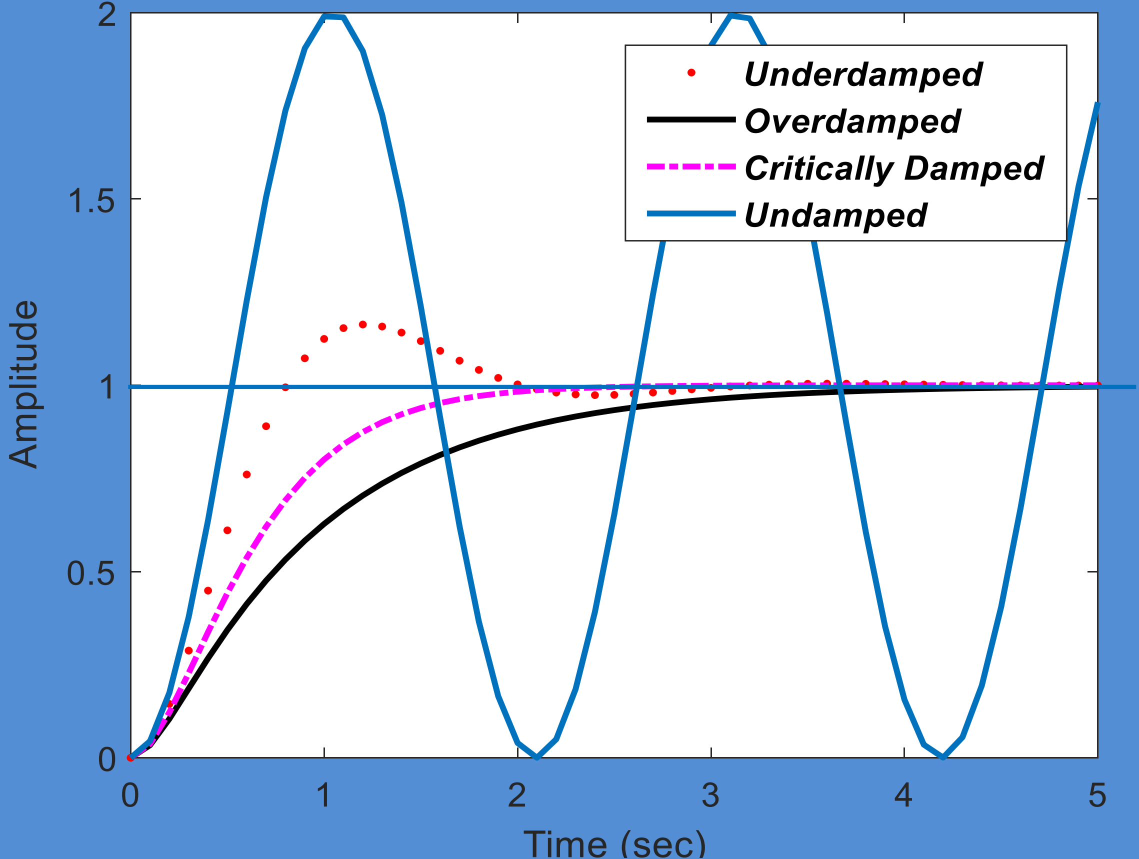
Critically Damped Response

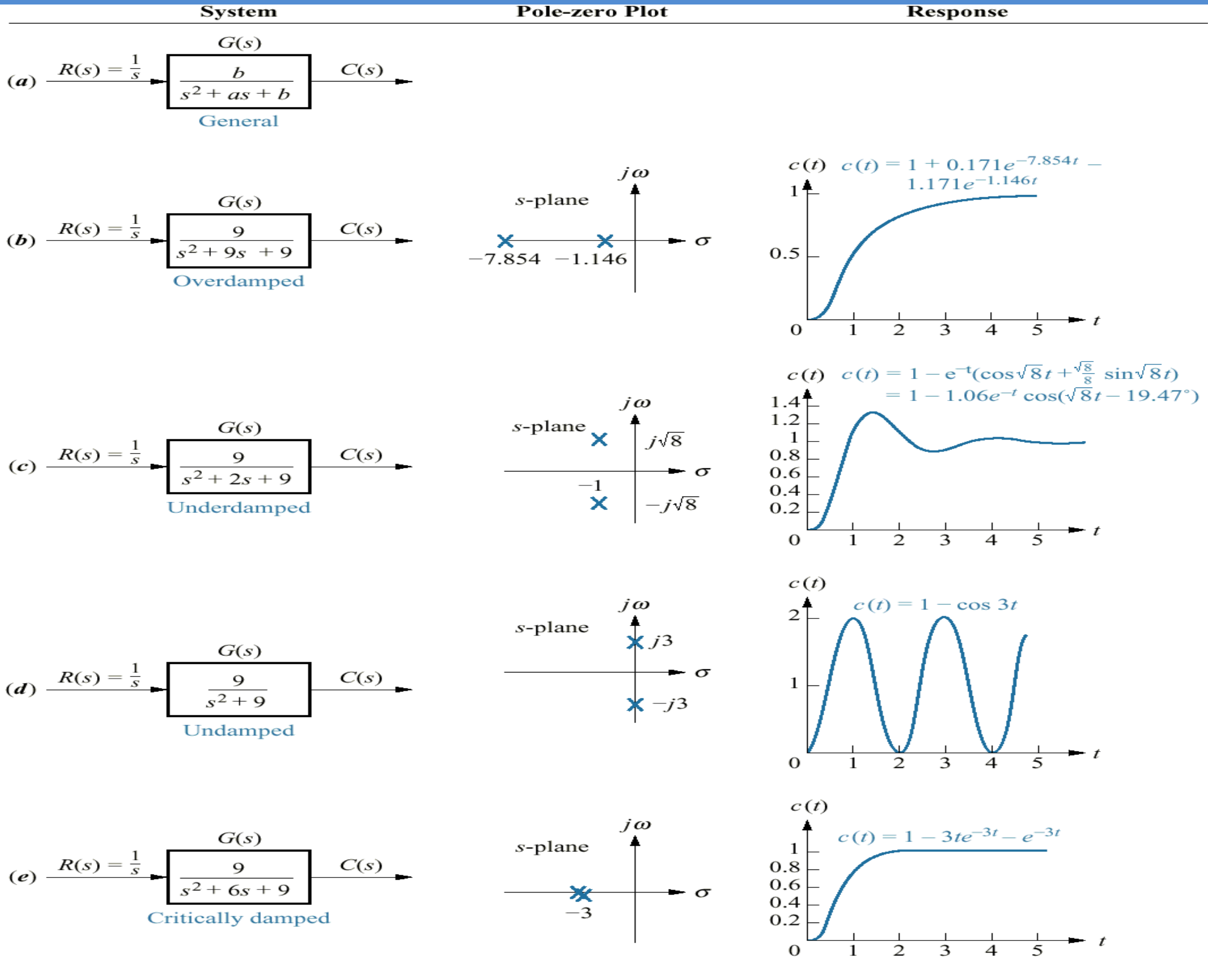


Matlab Program

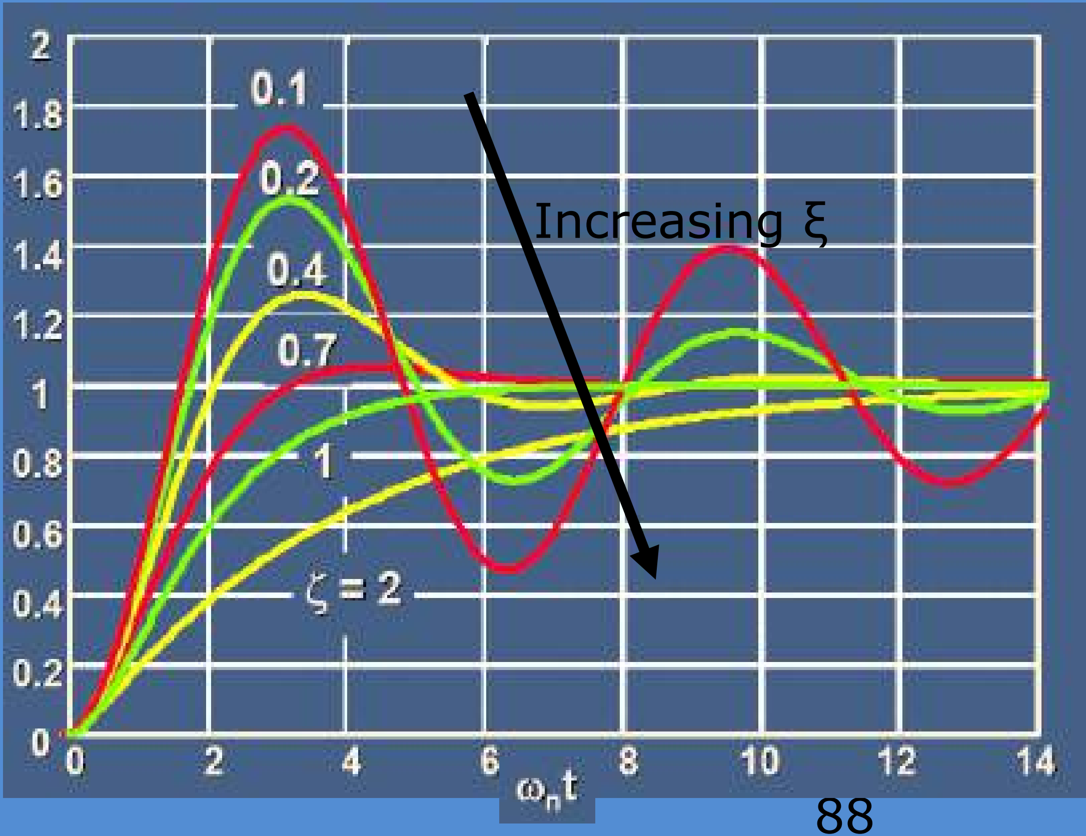
```
% Discuses the affect of damping ratio on control system
response
N1=[9];
d1=[1 3 9];
d2=[1 9 9];
d3=[1 6 9];
d4=[1 0 9];
 t=0:0.1:5;
y1= step(N1,d1,t);
y2= step(N1,d2,t);
y3= step(N1,d3,t);
y4= step(N1,d4,t);
 plot(t,y1,'.r',t,y2,'k',t,y3,'-.m',t,y4)
```

Second – Order System




System
Pole-zero Plot
Response
(a)
R(s) = 1/s
G(s)
b / (s^2 + as + b)
C(s)
General
(b)
R(s) = 1/s
G(s)
9 / (s^2 + 9s + 9)
C(s)
Overdamped
jω
s-plane
σ
−7.854
−1.146
c(t)
c(t) = 1 + 0.171e^{−7.854t} − 1.171e^{−1.146t}
1
0.5
0
1
2
3
4
5
t
(c)
R(s) = 1/s
G(s)
9 / (s^2 + 2s + 9)
C(s)
Underdamped
jω
s-plane
j√8
σ
−1
−j√8
c(t)
c(t) = 1 − e^{−t}(cos √8 t + (√8/8) sin √8 t)
= 1 − 1.06e^{−t} cos(√8 t − 19.47°)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
t
(d)
R(s) = 1/s
G(s)
9 / (s^2 + 9)
C(s)
Undamped
jω
s-plane
j3
σ
−j3
c(t)
c(t) = 1 − cos 3t
2
1
0
1
2
3
4
5
t
(e)
R(s) = 1/s
G(s)
9 / (s^2 + 6s + 9)
C(s)
Critically damped
jω
s-plane
σ
−3
c(t)
c(t) = 1 − 3te^{−3t} − e^{−3t}
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
t

Effect of different damping ratio, ξ



1. Response becomes faster and faster as the roots moved along the –ve real axis. The time constant $\frac{1}{\sigma}$ also decreases progressively.
2. Damping increases as the roots moves away in the –ve real dirction.
3. Frequency of oscillation increases as the roots move away from the real axis (along the imaginary axis direction).

All control system design methods attempt to shift the roots of the characteristic equation from an undesirable location to a desirable location.

5.5. Parameters of transient-response.

In many practical cases , the desired performance characteristics of control systems are specified in terms of time domain quantities .Systems with energy storage cannot respond instantaneously and will exhibit transient response whenever they are subjected to inputs or disturbances. Frequently , the performance characteristics of a control system are specified in terms of the transient response to a unit-step input since it is easy to generate and is sufficiently drastic.(If the response to a step input is known , it is mathematically possible to compute the response to any input).

The transient response of a system to a unit-step depends on the initial conditions. For convenience in comparing transient responses of various systems, it is a common practice to use the standard initial condition that the system is at rest initially with output and all time derivatives thereof zero. Then the response characteristics can be easily compared.

The transient response of a practical control system often exhibits damped oscillations before reaching steady state . In specifying the transient-response characteristics of a control system to a unit-step input , it is common to specifying the following:

1. Delay time , t_d

2. Rise time , t_r

3. Peak time , t_p

4. Maximum overshoot , M_p

5. Settling time , t_s

These specifications are defined in what follows and are shown graphically in fig.

1. Delay time, t_d : the delay time is the time required for the response to reach half the final value the very first time.

2. Rise time , t_r :the rise time is the time required for the response to rise from 10% to 90% ,5% to 95% , or 0% to 100% of its final

value. For under-damped second –order systems, the 0% to 100% rise time is normally used. For over-damped systems, the 10% to 90% rise time is commonly used.

3. Peak time, t_p: the peak time is the time required for the response to reach the first peak of the overshoot.

4. Maximum (percent) overshoot, M_p : the maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent over-shoot. It is defined by

$$Maximum\ percent\ overshoot = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

6. Settling time , t_s:the settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system . Which percentage error criterion to use may be determined from the objectives of the system design in question. The time-domain specifications just given are quite important since most control systems are time-domain systems; that is , they must exhibit acceptable time responses. (This means that the control system must be modified until the transient response is satisfactory). Note that if we specify the values of t_d, t_r, t_p, t_s and M_p , then the shape of the response curve is virtually determined. This may be seen clearly from Fig.5.3. Second-order systems and transient-response specifications. In the following , we shall obtain the rise time, peak time , maximum overshoot, and settling time of the second-order system given by Equation below. These values will be obtained in terms of ζ and ω_n.The system is assumed to be under-damped.

Rise time t_r: Referring to Equation, we obtain the rise time t_r by letting $c(t_r) = 1$ or

$$c(t_r) = 1 = 1 - e^{-\zeta w_n t_r}\left(\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r\right)$$

Since $e^{-\zeta w_n t_r} \neq 0$, we obtain from Equation the following equation:

$$\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r = 0$$

Or $\tan \omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$

Thus ,the rise time t_r is

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d}$$

Where β is defined in fig. clearly for a small value of t_r ,ω_d must be large.

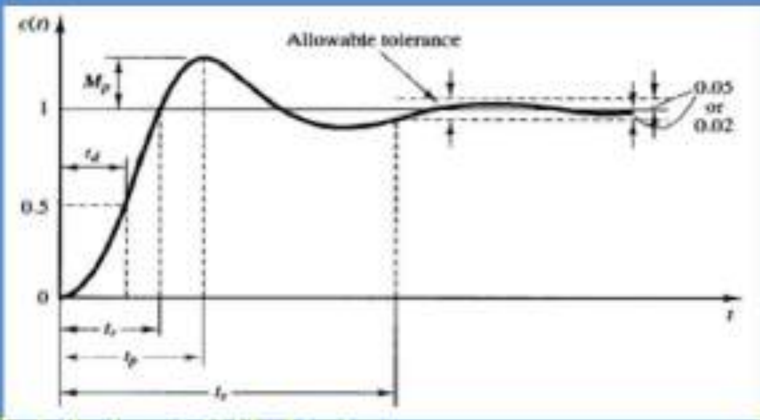


Fig. 5.4. Transient Response Parameters

Peak time , t_p: Referring to Equation , we may obtain the peak time by differentiating c(t)

with respect to time and letting this derivative equal can be simplified to

$$\frac{dc}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}(cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t)$$

$$+e^{-\zeta\omega_n t}(\omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t)$$

and the cosine terms in this last equation cancel each other , dc/dt , evaluated at $t = t_p$

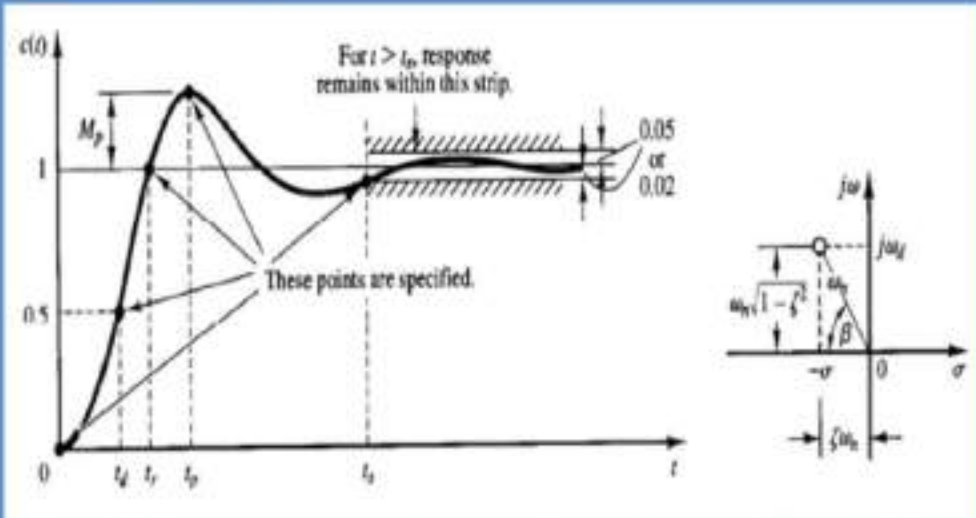


Fig. 5.5. Transient – response specifications and definition of the angle β

$$\left.\frac{dc}{dt}\right|_{t=t_p} = (\sin \omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$$

This last equation yields the following equation :

$$\sin \omega_d t_p = 0$$

Or $\omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$

Since the peak time corresponds to the first peak overshoot,

$\omega_d t_p = \pi$. Hence

$$t_p = \frac{\pi}{\omega_d}$$

The peak time t_p corresponds to one-half cycle of the frequency of damped oscillation.

Maximum overshoot M_p: The maximum overshoot occurs at the peak time or at $t = t_p = \pi/\omega_d$. Thus , M_p is obtained as

$$M_p = c(t_p) - 1 = -e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right.$$

$$= -e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

The maximum percent overshoot is $e^{-(\sigma/\omega_d)\pi} \times 100\%$.

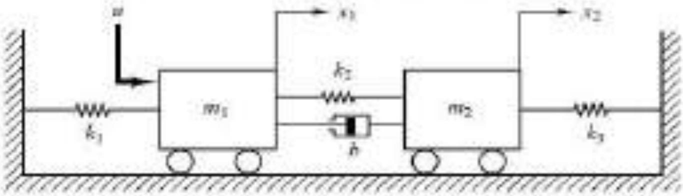
Settling time t_s: For an under-damped second-order system, the transient response is

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta^2}\right) \qquad , for\ t \geq 0$$

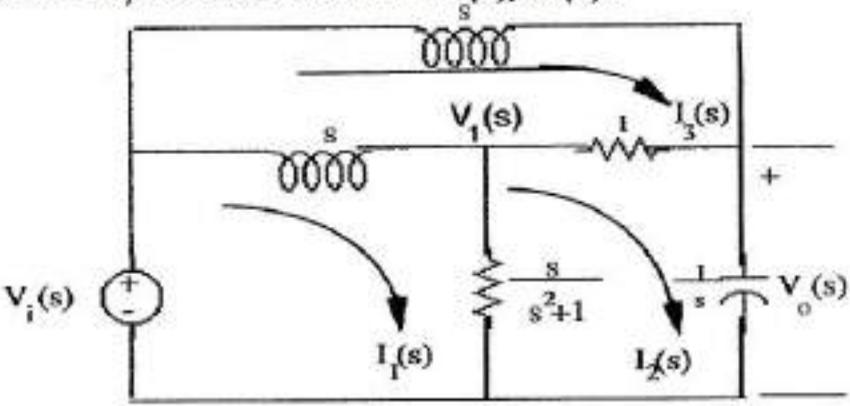
The curves $1 \pm (e^{-\zeta\omega_n t}\sqrt{1-\zeta^2}/)$ are the envelope curves of the transient response for a unit-step input. The response curve c(t) always remains within a pair of the envelope curves , as shown in Fig. below. The time constant of these envelope curves is $1/\zeta\omega_n$. .

Quiz No Three

Q1.A. For thefollowing mechanical system obtain the transfer function X1(s)/U(s)?



Q1.B. For the following electrical system, fine the state space representation of Vo(s)/Vi(s)?



Damped natural frequenc $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Damping ratio and natural frequency : $\sigma = \zeta\omega_n$

1. Delay time , t_d → $0.5\ t_r$
2. Rise time , t_r → $t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi-\beta}{\omega_d}$
3. Peak time , t_p → $t_p = \frac{\pi}{\omega_d}$
4. Maximum overshoot , M_p → $e^{-(\sigma/\omega_d)\pi} \times 100\%$
5. Settling time , t_s OR
   - $t_s = \frac{4}{\sigma}$ — 2 %
   - $t_s = \frac{3}{\sigma}$ — 5 %

The speed of decay of the transient response depends on the value of the time constant $1/\zeta\omega_n$. For a given ω_n , the setting time t_s is a function of the damping ratio ζ.

5.6. Solved problem

Prob.1. A field controlled d.c motor is characterized by the following differential equation.

$$0.5\frac{dw(t)}{dt}+w(t)=1.57i_f(t)$$

Where, w(t) is the angular velocity of the motor in radians/second and if is the field current in mA.

a)if the motor is supplied with a step input of 100mA what is the steady state speed in r.p.m.

at S.S ⟹ w=0

$$w_{ss} = 1.57 * 100 = 157 \frac{rad}{second} = 157 \frac{60}{2\pi} \, r.p.m$$

$$= 1499.23 \, r.p.m$$

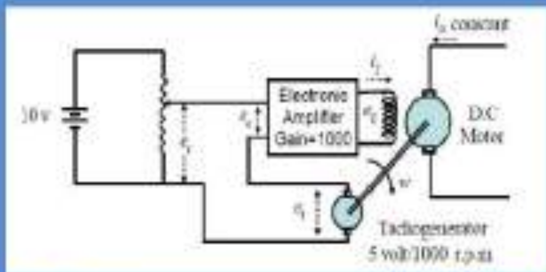
b)in (a) how much time would be taken by the motor to reach i)25% , ii)50% and iii)75% of the steady state speed?

Characteristic equation

$(0.5m+1)=0$

$m = -2$

$w_{Tr} = Ae^{-2t}$



At t=0 , w(0)=0

0=157+A

A=−157

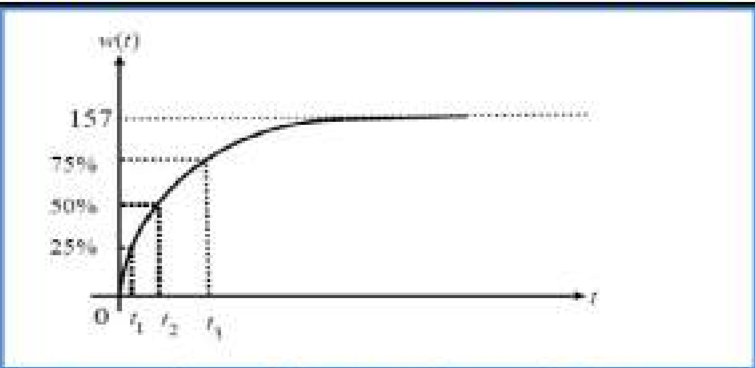
$w(t) = 157 * (1 - e^{-2t})$

i) $w(t_1)$ = 25% of the S.S speed (157 rad/second)

$\frac{25}{100} * 157 = 157 * (1 - e^{-2t_1}) \Rightarrow \qquad t_1 = 0.1438\, sec$

$\frac{50}{100} * 157 = 157 * (1 - e^{-2t_2}) \Rightarrow \qquad t_2 = 0.3466\, sec$

$\frac{75}{100} * 157 = 157 * (1 - e^{-2t_3}) \Rightarrow \qquad t_3 = 0.6931\, sec$



c)The above motor is used in a speed control scheme as shown in figure below. Draw the block diagram of the system and write down the differential equation of the closed loop system. Given that field resistance =100Ω, inductance 20H.

i) M

$$e_f = R_f i_f + L_f \frac{di_f}{dt}$$

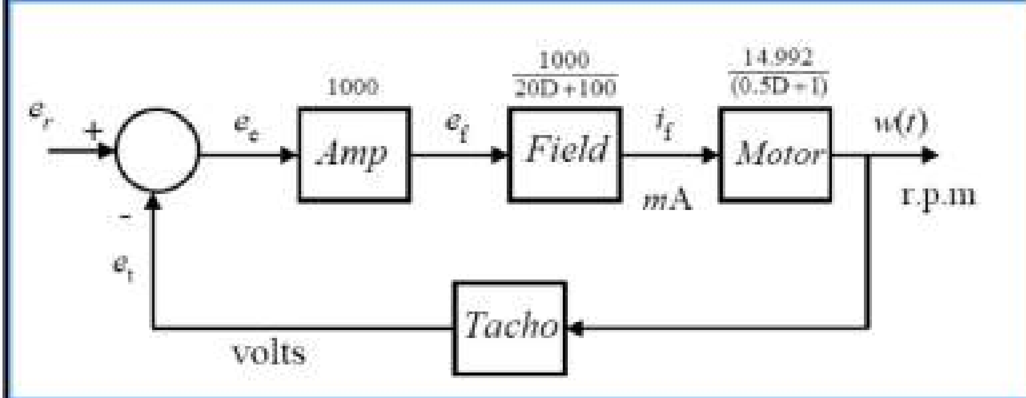
$$e_f = 100 * i_f + 20Di_f$$

$$\frac{i_f}{e_f} = \frac{1}{100+20D} = \frac{1000}{100+20D} mA$$

from system equation.

$$(0.5\,D + 1)w = 1.57i_f$$

$$\frac{w}{i_f} = \frac{1.57}{0.5D+1} \text{ in rad/second } = \frac{14.992}{0.5\,D+1} \text{ in r.p.m}$$



d) calculate the setting of the potentiometer to get a steady state speed of

i) 900 r.p.m

ii) 1100 r.p.m

$$G = 1000 * \frac{1000}{20D + 100} * \frac{14.992}{0.5D + 1} = \frac{1499200}{(D+2)(D+5)}$$

H=0.005 volt/r.p.m

$$\frac{w(t)}{e_r(t)} = \frac{G}{1 + GH} = \frac{1499200}{D^2 + 7D + 7506}$$

$$\frac{d^2w(t)}{e_r(t)} + 7\frac{dw(t)}{dt} + 7506\, w(t) = 1499200\, e_r(t)$$

Differential Equation of the C.L system

i) For $w(t)|_{t=\infty} = 900\ r.p.m$

$D = D^2 = 0$ at steady state

$$w(t)_{S.S} = 900 = e_r \frac{1499200}{7506}$$

$e_r = 4.506$ volts

Potentiometer factor =0.4506

ii) For $w(t)_{S.S} = 1100$ r.p.m $\qquad e_r = 5.507$ volts

Potentiometer factor =0.5507

if the amplifier gain suddenly decreases by 25% what would be the range in the motor speed if it was earlier running at 900 r.p.m. when the motor is running at 900 r.p.m

$e_r = 4.506$ volts

Amplifier gain=750

$$\frac{w(t)}{e_r(t)} = \frac{1124400}{D^2 + 7D + 5632}$$

At S.S $w(t)|_{t=\infty} = \frac{4.506*1124400}{5632} = 899.6$ r.p.m

Prob.2. A small electric oven is known to have a first order differential equation as its describing equation. when the rated input of 20 volt is applied to the oven at 25°C, the steady state temperature is found to be 1225°C and a temperature of 625°C is reached in 30 seconds.

a)Write down the differential equation of the oven.

General first order differential equation.

$$\frac{dT(t)}{dt} + aT(t) = be_1(t)$$

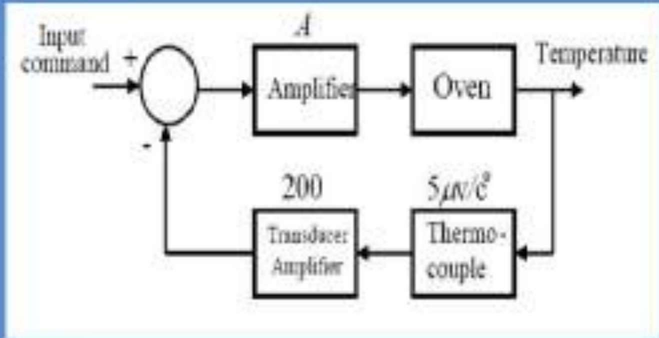
Oven Oven

Temperature Voltage

t=0 , T=25°

t=30 , T=625°

t=∞ , T=1225°



$$T_{ss} = \frac{b}{a} e_1 \quad ; T_{tr} = Ae^{-at}$$

$$T_{total} = Ae^{-at} + \frac{b}{a} e_1$$

Initial condition at t=0 , T(t)=25

$$25 = A + \frac{b}{a}e_1 \ , \qquad A = 25 - \frac{b}{a}e_1$$

$$T(t) = \left(25 - \frac{b}{a}e_1\right) * e^{-at} + \frac{b}{a}e_1$$

At t=∞ (steady state); T(t)=1225°C

$$1225 = \frac{b}{a}20$$

At t=30 , T(t)=625

$$625 = \left(25 - \frac{b}{a}20\right) * e^{-30a} + \frac{b}{a}20$$

$$625 = (25 - 1225) * e^{-30a} + 1225$$

a=0.0231049

b=1.4151755

Oven equation is

$$\frac{dT(t)}{dt} + 0.023T(t) = 1.415e_1(t)$$

b) it is now required to control the temperature of the oven by a close loop feedback system as shown in figure below. Obtain the differential equation of the overall system.

$$G = A\frac{1.415}{D + 0.023} \;;H = 5 * 10^{-6} * 200 = 10^{-3}$$

$$\frac{T(t)}{e_1} = \frac{G}{1 + GH} = \frac{1.415A}{D + 0.023 + A * 1.415 * 10^{-3}}$$

c)calculate the value of 'A' such that if 'A' increases by 10% the steady state change in the oven temperature does not exceed 0.5°C for $e_1 = 1$ volts

$$T_1 - T_2 = 0.5$$

$$\frac{1.415 * 1.1 * A}{0.023 + 1.415 * 1.1 * 10^{-3} * A} - \frac{1.415 * A}{0.023 * 1.415 * 10^{-3} * A} = 0.5$$

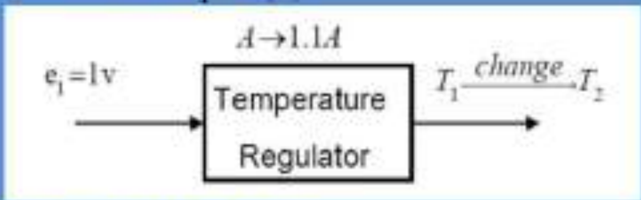
$$1.10122375 * A^2 + 34.172251 * A + 264.5 = 3254.5 * A$$

$$1.10122375 * A^2 - 3220.32775 * A + 264.5 = 0$$

$$A = \frac{3220.3277 \pm \sqrt{(3220.3277)^2 - 4 * 1.10122375 * 264.5}}{2 * 1.10122375}$$

A=2924.158

d) Calculate the time constant of the closed loop system for the value of 'A' calculated in part (c).



$$\frac{T(t)}{e_1} = \frac{1.415A}{D + 0.023 + A * 1.415 * 10^{-3}} = \frac{K}{(D + a)}$$

$$a = 0.023 + 2924.153 * 1.415 * 10^{-3} = 4.160676$$

Time constant= $T = \frac{1}{a} = 0.240345\ sec$

e)what is the range of input command in volts required for controlling the temperature from 100°C to 1000°C.

At $S.S \quad T = \frac{1.415*2924.153}{0.023+2924.153*1.415*10^{-3}} e_1 = \frac{4137.676496}{4.1606764} e_1$

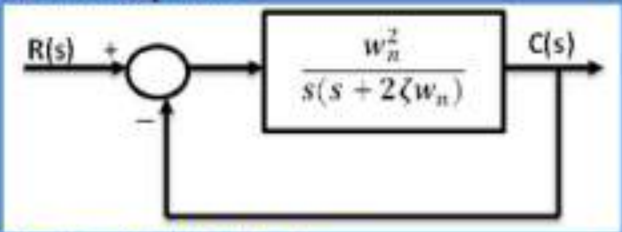
$$T = 994.472 * e_1$$

at T=100 : 100=994.472*e_1 , $e_1 = 0.100555$ volt

at T=1000 : $1000 = 994.472 * e_1$, $e_1 = 1.00555$ volt

The range of input command is $0.100555 \le e_1 \le 1.00555$

Prob. 3: .For the system shown in Fig. below where $\zeta = 0.6$ and $\omega_n = 5\, rad/sec$. Let us obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time t_s when the system is subjected to a unit-step unit.



Solution:

From the given values of ζ and ω_n , we obtain

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$ and $\sigma = \zeta\omega_n = 3.$

Rise time t_r: The rise time is

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

where β is given by

$$\beta - \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93\, rad$$

The rise time t_r is thus :

$$t_r = \frac{3.14 - 0.93}{4} = 0.55\, sec$$

peak time t_p: The peak time is

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785\ sec$$

Maximum over shoot M_p:

The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.025$$

The maximum percent overshoot is thus 9.5%

Settling time t_s: For the 2% criterion , the settling time is

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33\ sec$$

For the 5% criterion

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1\ sec$$

University of Anbar
College of Engineering
Dept. of Electrical Engineering

Control Theory I
Assist. Prof. Dr. Yousif Al Mashhadany
2019 - 2020

Lecture No. Six

Steady-State Error

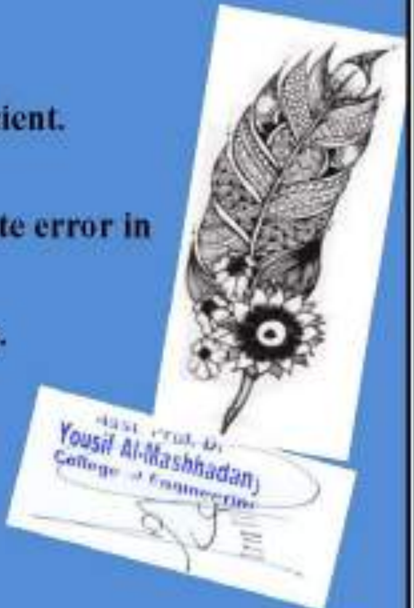
This lecture will discusses the following topics

6.1. **Introduction.**

6.2. **Steady-State Step Error Coefficient.**

6.3. **Comparison between steady state error in open loop & closed loop system.**

6.4. **Solved problems**



6.1. Introduction.

The simple closed-loop feedback system, with unity feedback, shown in Fig. 6.1, may be called a tracker since the output c(t) is expected to track or follow the input r(t). The open-loop transfer function for this system is ($G(s)=C(s)/E(s)$),which is determined by the components of the actual control system. Generally G(s) has one of the following mathematical forms:

$$\left.\begin{aligned} G(s) &= \frac{K_o(1+T_1 s)(1+T_2 s)\ldots\ldots}{(1+T_a s)(1+T_b s)\ldots\ldots} \\ G(s) &= \frac{K_1(1+T_1 s)(1+T_2 s)\ldots\ldots}{s(1+T_a s)(1+T_b s)\ldots\ldots} \\ G(s) &= \frac{K_2(1+T_1 s)(1+T_2 s)\ldots\ldots}{s^2(1+T_a s)(1+T_b s)\ldots\ldots} \end{aligned}\right\} \qquad (6.1)$$

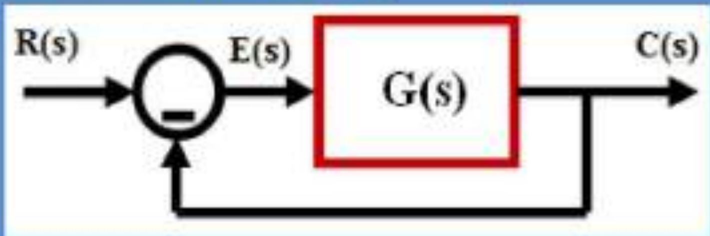


Fig. 6.1. Unity–feedback control system

Note that the constant term in each factor is equal to unity. The preceding equations are expressed in a more generalized manner by defining the standard form of the transfer function as:

$$G(s)=\frac{K_m(1+b_1s+b_2s^2+b_ws^w)......}{s^m(1+a_1s+a_2s^2+a_us^u).....}=K_mG'(s) \qquad (6.2)$$

Where

$a_1, a_2, ...$=constant coefficients

$b_1, b_2, ...$=constant coefficients

K_m = gain constant of the transfer function G(s)

m = 0,1,2,.... Denotes the transfer function type

G'(s) = forward transfer function with unity gain

The degree of the denominator is n=m u. For a unity-feedback system, E and C have the same units. Therefore, K_0 is non-dimensional, K_1 has the units of seconds-1, K_2 has the units of seconds-2.

In order to analyze each control system, a "type" designation is introduced. The designation is based upon the value of the exponent m of s in Equation(6.1). Thus, when m=0, the system represented by this equation is called a Type 0 system; when m=1, it is called a Type 1 system; when m=2, it is called a Type 2 system; etc. Once a physical system has been expressed mathematically, the analysis is independent of the nature of the physical system. It is immaterial whether the system is electrical, mechanical, hydraulic, thermal, or a combination of these. The most common feedback control systems have Type 0, 1, or 2 open-loop transfer functions. It is important to analyze each type thoroughly and to relate it as closely as possible to its transient and steady-state solution. The various types exhibit the following steady-state properties:

Type 0: A constant actuating signal results in a constant value for the controlled variable.

Type 1: A constant actuating signal results in a constant rate of change (constant velocity) of the controlled variable.

Type 2: A constant actuating signal results in a constant second derivative (constant acceleration) of the controlled variable.

Type 3: A constant actuating signal results in a constant rate of change of acceleration of the controlled variable.

These classifications lend themselves to definition in terms of the differential equations of the system and to identification in terms of the forward transfer function. For all classifications the degree of the denominator of G(s)H(s) usually is equal to or greater than the degree of the numerator because of the physical nature of feedback control systems. That is, in every physical system there are energy-storage and dissipative elements such that there can be no instantaneous transfer of energy from the input to the output. However, exceptions do occur.

6.2. Steady-State Step Error Coefficient.

The error coefficients are independent of the system type. They apply to any system type and are defined for specific forms of the input, i.e., for a step, ramp, or parabolic input. These error coefficients are applicable only for stable unity feedback systems. The results are summarized in Table 6.1.

Table 6.1. Definitions of Steady-State Error Coefficients for Stable Unity- Feedback Systems

Error coefficient	Definition of error coefficient	Value of error coefficient	Form of input signal r(t)
Step (K_p)	$\frac{c(t)_{ss}}{e(t)_{ss}}$	$\lim_{s\to 0} G(s)$	$R_0 u_{-1}(t)$
Ramp (K_v)	$\frac{(Dc)_{ss}}{e(t)_{ss}}$	$\lim_{s\to 0} sG(s)$	$R_1 t u_{-1}(t)$
Parabolic(K_a)	$\frac{(D^2c)_{ss}}{e(t)_{ss}}$	$\lim_{s\to 0} s^2G(s)$	$\frac{R_2 t^2 u_{-1}(t)}{2}$

The step error coefficient is defined as:

$$step\ error\ coefficient = \frac{steadystate\ value\ of\ output\ c(t)_{ss}}{steadystate\ value\ of\ acuating\ signal\ e(t)_{ss}} = K_s$$

and implies only for a step input, $r(t)=R_0 u_{-1}(t)$.the steady state value of the output is obtained by apply final value theorem.

$$C(t)_{ss} = \lim_{s\to 0} sC(s) = \lim_{s\to 0}[\frac{sG(s)}{1+G(s)}\frac{R_0}{s}] = \lim_{s\to 0}[\frac{G(s)}{1+G(s)}R_0]$$

Similarly for $e(t)_{ss}$

$$e(t)_{ss} = \lim_{s\to 0} sC(s) = \lim_{s\to 0}[s\frac{1}{1+G(s)}\frac{R_0}{s}] = \lim_{s\to 0}[\frac{1}{1+G(s)}R_0]$$

Substitute the above two equation to get step error coefficient

$$\text{step error coefficient} = \frac{\lim_{s\to 0}\left[\frac{G(s)}{1+G(s)}R_0\right]}{\lim_{s\to 0}\left[\frac{1}{1+G(s)}R_0\right]}$$

Since both numerator and denominator of the above equation in the limit can't be zero or infinity simultaneously, where $K_0 \neq 0$, the indeterminate $0/0$ or ∞/∞ never occur. Thus this equation reduces to K_p. Therefore applying (step error coefficient $= \lim_{s\to 0} G(s) = K_p$) to each type system yields

$K_p = \lim_{s\to 0} \frac{K_0(1+T_1 s)(1+T_2 s)\ldots\ldots}{(1+T_a s)(1+T_b s)\ldots\ldots} = K_0$ for type zero system

$K_p = \infty$ for type one system

$K_p = \infty$ for type two system

The ramp error coefficient is defined as:

$$Ramp\ error\ coefficient = \frac{steady state\ derivative\ of\ output\ Dc(t)_{ss}}{steady state\ value\ of\ acuating\ signal\ e(t)_{ss}} = K_v$$

Therefore applying (ramp error coefficient $= \lim_{s \to 0} sG(s) = K_v$) to each type system yields

$K_v = \lim_{s \to 0} s \frac{K_0(1+T_1 s)(1+T_2 s)......}{(1+T_a s)(1+T_b s)....} = 0$ for type zero system

$K_v = K_1$ for type one system

$K_v = \infty$ for type two system

The parabolic error coefficient is defined as:

$$Parabolic\ error\ coefficient =$$

$$\frac{steady state\ of\ second\ derivative of\ output D^2 c(t)_{ss}}{steady state\ value\ of\ acuating\ signal\ e(t)_{ss}} = K_a$$

Therefore applying (ramp error coefficient $= \lim_{s \to 0} s^2 G(s) = K_a$) to each type system yields

$$K_a = \lim_{s \to 0} s^2 \frac{K_0(1+T_1 s)(1+T_2 s)\ldots\ldots}{(1+T_a s)(1+T_b s)\ldots\ldots} = 0$$ for type zero system

$K_a = 0$ for type one system

$K_a = K_2$ for type two system

Table 6.2. below gives the values of the error coefficients for the Type 0,1, and 2 systems. These values are determined from Table 6.1. The reader should be able to make ready use of Table 6.2 for

evaluating the appropriate error coefficient. The error coefficient is used with the definitions given in Table 6.1 to evaluate the magnitude of the steady-state error.

Table 6.2. Steady-State Error Coefficients for Stable Systems

System type	Step error coefficient K_p	Ramp error coefficient K_v	parabolic error coefficient K_a
0	K_0	0	0
1	∞	K_1	0
2	∞	∞	K_2

6.3. Comparison of steady state errors in open loop and closed loop systems:

Consider the open loop control system and closed loop control system shown in Fig. 6.2(a,b).

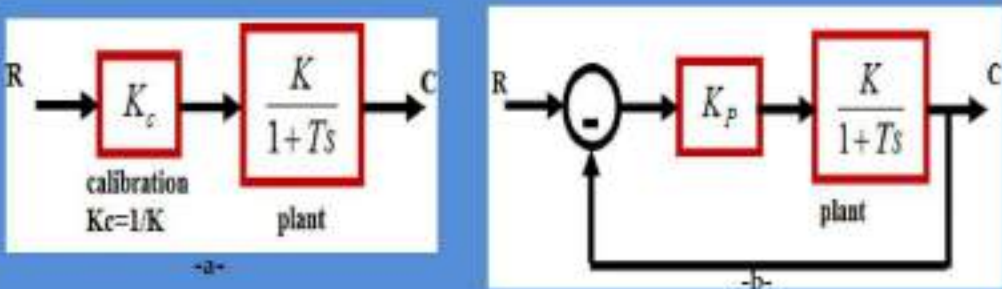


Fig.6.2. (a) block diagram of open loop, (b) closed loop system

In the open loop control system the gain Kc is calibrated so that $Kc=1/K$, thus the transfer function of open loop control system is:

$$G_0(s)=\frac{1}{K}\frac{K}{1+sT}=\frac{1}{1+sT}$$

In the closed loop control system the gain K_p of the controller is set so that $K_pK>>1$

Assuming a step input, let us compare the steady state errors for those control systems. For the open loop control system the error signal is:

$e(t)=r(t)-c(t)$; Or

$$E(s)=R(s)-C(s); \rightarrow E(s)=[1-G_0(s)]R(s)$$

The steady state error in a unit step response is

$e_{ss} = \lim_{s \to 0} sE(s)$; $e_{ss} = \lim_{s \to 0} s[1 - G_0(s)]\frac{1}{s}$; $\rightarrow = 1 - G_0(0)$

Table 6. 3. Steady state error in term of gain K

	Step input r(t)=1	ramp input r(t)=t	parabolic input r(t)=1/2 t^2
Type 0 system	$\frac{1}{1+k}$	∞	∞
Type 1 system	0	$\frac{1}{k}$	∞
Type 2 system	0	0	$\frac{1}{k}$

If the $G_o(0)$,dc gain of the open loop control system is equal to unity, then the steady state error is zero. Due environmental change and aging of the components, however the dc gain $G_o(0)$ will drift from unity as time elapses and steady state error will no longer be equal to zero. Such steady state error will remain until the system is recalibrated see table (6.3).

For the closed loop control system the error signal is:

$$E(s) = R(s) - C(s)$$

$$E(s) = [\frac{1}{1+G(s)}]R(s)$$

Where : $G(s) = \frac{KK_p}{1+Ts}$

The steady state error in the unit step response is

$$e_{ss} = \lim_{s\to 0} s\left[\frac{1}{1+G(s)}\right]\frac{1}{s}$$

$$= \frac{1}{1+G(0)} = \frac{1}{1+K_p K}$$

In the closed loop system, gain K_p is set at a large value compared to 1/K. Thus the steady state error can be made small but not exactly zero.

Let us assume the following variation in the transfer function of the plant, assuming Kc and K_p constant.

$$\frac{K+\Delta K}{1+Ts}$$

As an example let us assume that K=10,ΔK=1 or ,$\Delta K/K$=0.1. Then the steady state error in the unit step response becomes

$$e_{ss}=1-\frac{1}{K}(K+\Delta K)=1-1.1=-0.1$$

For the closed loop system, if gain K_p is set at $100/K$, then the steady state error in the unit step response becomes

$$e_{ss}=\frac{1}{1+G(0)}$$

$$e_{ss}=\frac{1}{1+\frac{100}{k}(K+\Delta K)}$$

$$e_{ss}=\frac{1}{1+110}=0.009$$

Thus closed loop control system is superior to open loop control system in the presence of environmental changes, aging of components and the like, which definitely affect the steady state performance.

6.4. Solved problems

Prob. 1. A closed loop system as a forward transfer function given by:

$$G(s)=\frac{k}{2s^2+16s+16},H(s)=\frac{1}{s}$$

Evaluate steady state error for input (r(t)=2+t)); when the gain is equal to (2)?

Solution:

Open loop T.F.= G(s)*H(s)= $\dfrac{k}{s(2s^2+16s+16)}$

$R(s)=\dfrac{2}{s}+\dfrac{1}{s^2}$.

$e_{ss}=e_{ss\,position}+e_{ss\,velocity}$

$e_{ss\ position}=\dfrac{A}{1+Kp}, A=\text{constant}$ **(r(t)=2+t)=r(t)=A+Bt+ct^2; A=2 and B=1**

$K_p=\lim_{s\to 0}G(s)=\lim\dfrac{K}{s(2s^2+16s+16)}=\ \dfrac{\mathbf{2}}{\mathbf{0}}=\infty$

So that

$e_{ss\ position}=\dfrac{2}{1+\infty}=0, A=2$

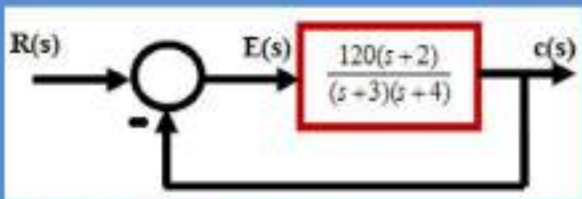
$$K_v = \lim_{s\to 0} G(s) = \lim s \frac{K}{s(2s^2+16s+16)} = 1/4, K = 2$$

$$e_{ss\ velocity} = \frac{B}{Kv} = \frac{1}{1/4} = 4, B = 1$$

$$e_{ss} = e_{ss\ velocity} + e_{ss\ position} = 4+0=4$$

Note; these two values cannot be added where everyone represents system response for certain input.

Prob. 2. Find the steady state errors for the inputs, $5u(t)$, $5tu(t)$ *and* $5t^2u(t)$ to the system shown below, the function $u(t)$ is the unit step.



For the input $5u(t)$,the Laplace transform is 5/s, the steady state error will be :

$$e(\infty)=e_{step}(\infty)=\frac{5}{1+\lim_{s\to 0}G(s)}=\frac{5}{1+\frac{120*2}{3*4}}=\frac{5}{1+20}=\frac{5}{21}$$

For the input $5tu(t)$,the Laplace transform is $5/s^2$, the steady state error will be :

$$e(\infty)=e_{ramp}(\infty)=\frac{5}{\lim_{s\to 0}sG(s)}=\frac{5}{0}=\infty$$

For the input $5t^2u(t)$,the Laplace transform is $10/s^3$, the steady state error will be :

$$e(\infty)=e_{parabola}(\infty)=\frac{10}{\lim_{s\to 0}s^2G(s)}=\frac{10}{0}=\infty$$

Example 3.A unity feedback system has the following forward transfer function:

$$G(s)=\frac{1000(s+8)}{(s+7)(s+9)}$$

Use Matlab to find $K_p, e_{step}(\infty)$ and the closed loop poles to check the stability for the system.

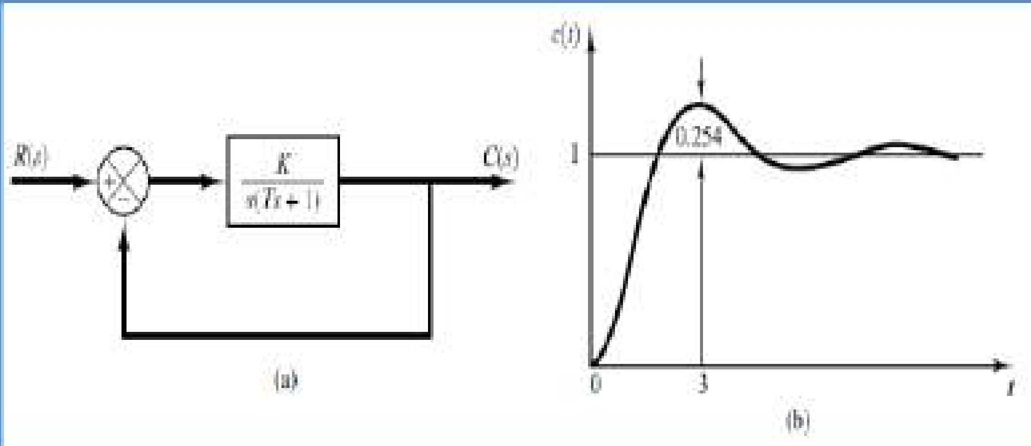
```
numg=1000*[1 8];
deng=poly([-7 -9]);
G=tf (numg,deng);
Kp=dcgain(G)
Estep=1/(1+Kp)
T=feedback(G,1); poles=pole(T)
```

H.W. Find the value of K to yield a 10% error in the steady state for a unity feedback who has the following forward transfer function. Try to write Matlab code to solve this problem.

$$G(s) = \frac{K(s+12)}{(s+14)(s+18)}$$

Answer: K=189

Example (4): When the system shown in Figure (a) is subjected to a unit-step input, the system output responds as shown in Figure (b). Determine the values of K and T from the response curve?



Solution: The maximum overshoot of **25.4%** corresponds to **ζ=0. 4.** From the response curve we have

$$t_p = 3 \qquad t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n\sqrt{1-0.4^2}} = 3$$

It follows that $\omega_n = 1.14$ From the block diagram we have $\frac{C(s)}{R(s)} = \frac{K}{Ts^2 + s + K}$

$$\omega_n = \sqrt{\frac{K}{T}}, \qquad 2\zeta\omega_n = \frac{1}{T}$$

Therefore, the values of T and K are determined as

$$T = \frac{1}{2\zeta\omega_n} = \frac{1}{2 \times 0.4 \times 1.14} = 1.09$$

$$K = \omega_n^2 T = 1.14^2 \times 1.09 = 1.42$$

Quiz No. Four

Q1. Consider the system shown in Figure 1. Determine the value of k such that the damping ratio ζ is 0.5.Then obtain the rise time t_r , peak time t_p, maximum overshoot M_p, and settling time t_s in the unit-step response?

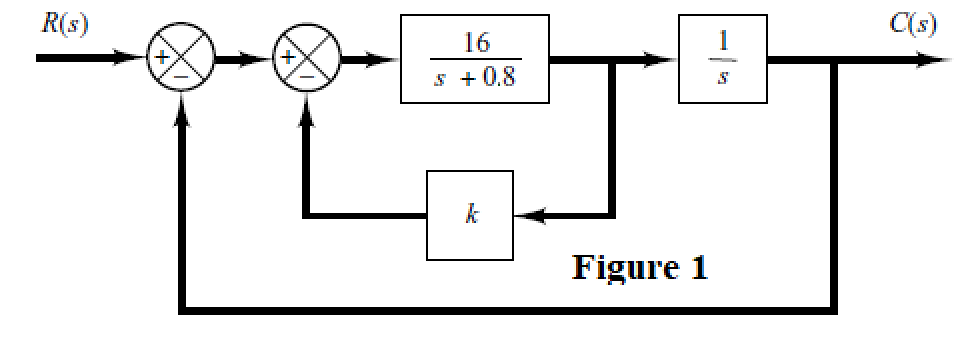


Figure 1

University of Anbar
College of Engineering
Dept. of Electrical Engineering

Control Theory I
Assist. Prof. Dr. Yousif Al Mashhadany
2019 - 2020

Lecture No. Seven

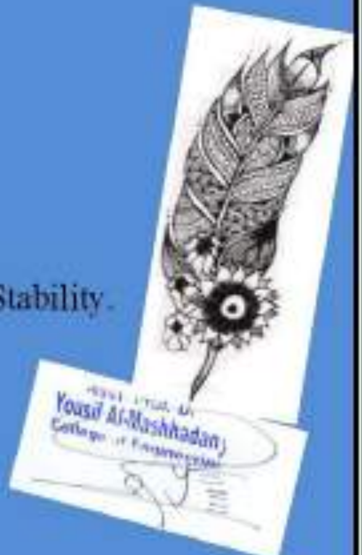
Routh's Stability Criterion

This lecture discusses the topics:

7.1. Introduction.

7.2. Routh's Criteria Rules.

7.3. Solved problem for Checking System Stability.



7.1. Introduction.

The response transform $X_2(s)$ has the general form given by Equation (7.1),which is repeated here in slightly modified form. $X_1(s)$ is the driving transform.

$$X_2(s)=\frac{P(s)}{Q(s)}X_1(s)=\frac{P(s)X_1(s)}{b_n s^n+b_{n-1}s^{n-1}+\dots\dots+b_1 s+b_0} \qquad (7.1)$$

The stability of the response $X_2(t)$ requires that all zeros of Q(s) have negative real parts. Since it is usually not necessary to find the exact solution when the response is unstable, a simple procedure to determine the existence of zeros with positive real parts is needed. If such zeros of Q(s) with positive real parts are found, the system is unstable and must be modified. Routh's criterion is a simple method of determining the number of zeros with positive real parts without actually solving for the zeros of Q(s). Note that zeros of Q(s) are poles of $X_7(s)$. The characteristic equation is

$$Q(s) = b_n s^n + b_{n-1} s^{n-1} + \dots\dots + b_1 s + b_0 = 0 \qquad (7.2)$$

If the bo term is zero, divide by s to obtain the equation in the form of Equation (7.2). The b's are real coefficients, and all powers of s from s^n to s^0 must be present in the characteristic equation. A necessary but not sufficient condition for stable roots is that all the coefficients in Equation (7.2) must be positive. If any coefficients other than b_0 are zero, or if all the coefficients do not have the same sign, then there are pure imaginary roots or roots with positive real

7.2. Routh's Criteria Rules:

parts and the system is unstable. In that case it is unnecessary to continue if only stability or instability is to be determined. When all the coefficients are present and positive, the system may or may not be stable because there still may be roots on the imaginary axis or in the right-half s plane.

Routh's criterion is mainly used to determine stability. In special situations it may be necessary to determine the actual number of roots in the right half s plane. For these situations the procedure described in this section can be used.

The coefficients of the characteristic equation are arranged in the pattern shown in the first two rows of the following Routhian array. These coefficients are then used to evaluate the rest of the constants to complete the array.

S^n	b_n	b_{n-2}	b_{n-4}	b_{n-6}	…….
S^{n-1}	b_{n-1}	b_{n-3}	b_{n-5}	b_{n-7}	…….
S^{n-2}	C1	C2	C3	…….	
S^{n-3}	d1	d2	d3	……	
.	.				
.	.				
S	j1				
S^0	k1				

The constants c1, c2, c3, ... etc., in the third row are evaluated as follows:

$$C_1 = \frac{(b_{n-1})(b_{n-1}) - (b_{n-3})(b_x)}{b_{n-1}}$$

$$C_2 = \frac{(b_{n-1})(b_{n-4}) - (b_{n-5})(b_x)}{b_{n-1}}$$

$$C_3 = \frac{(b_{n-1})(b_{n-6}) - (b_{n-7})(b_x)}{b_{n-1}}$$

This pattern is continued until the rest of the c's are all equal to zero. Then the d row is formed by using the sn-1 and sn-2 rows. The constants are:

$$d_1 = \frac{C_1(b_{n-3}) - (b_{n-1})C_2}{C_1}$$

$$d_2 = \frac{C_1(b_{n-5}) - (b_{n-1})C_3}{C_1}$$

$$d_3 = \frac{C_1(b_{n-7}) - (b_{n-1})C_4}{C_1}$$

This process is continued until no more d terms are present. The rest of the rows are formed in this way down to the s0 row. The complete array is triangular, ending with the s0 row. Notice that the s1 and s0 rows contain only one term each. Once the array has been found, Routh's criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of changes of sign of the coefficients in the first column. Therefore, the system is stable if all terms in the first column have the same sign.

7.3. Solved problem for Checking System Stability.

Prob.1. Check the stability of the control system that it has characteristic equation in the following:

$Q(s)= s^5 + s^4 + 10s^3 + 72s^2 + 152s + 240$?

Solution:

The Routhian array is formed by using the procedure described above:

S^5	1	10	152
S^4	1	72	240
S^3	-62	-88	
S^2	70.6	240	
S	122.6		
S^0	240		

In the first column there are two changes of sign, from 1 to -62 and from 62 to 70.6; therefore, Q(s) has two roots in the right-half s plane (RHP).Note that this criterion gives the number of roots with positive real parts but does not tell the values of the roots. If the characteristic equation is factored, the roots are $s_1 = -3$, $s_{2,3} = -1 \pm j\sqrt{3}$, and $s_{4,5} = +2 \pm j\sqrt{4}$. This calculation confirms that there are two roots with positive real parts. The Routh criterion does not distinguish between real and complex roots.

Prob.2. Check the stability of the control system that it has the following characteristic equation(C.E):

C.E.= $s^2 + 3s + 2$?

Solution:

$$\begin{array}{lll} S^2 & 1 & 2 \\ S^1 & 3 & 0 \\ S^0 & \dfrac{3*2-0*1}{3}=2 & \end{array}$$

Because no change in the first column (pivoted column), there are no poles in the right hand side (RHS) and hence the system is stable.

Prob.3. Check the stability of the control system that it has clc's eqn. in the following:

$Q(s) = s^4 + 3s^3 + s^2 + 3s + 1$?

Solution: The routh's array:

S^4 1 1 1

S^3 3 3 0

S^2 0 1

S ? ?

S ?

This is one of the special cases ,so that when we get zero in routh's array to fill full this theory replace the zero by symbol (δ) and then can be determined the range of stability for this system:

$$
\begin{array}{llll}
S^4 & 1 & 1 & 1 \\
S^3 & 3 & 3 & 0 \\
S^2 & \delta & 1 & \\
S^1 & (3\delta-3)/\delta & 0 & \\
S^0 & 0 & &
\end{array}
$$

If we consider δ a very small positive number [it has either a very small positive or a very small negative and this is optional and both of them gives same final result]

$A=(3\delta-3)/\delta=3-3/\delta$

$\lim_{s \to 0} A = 3 - \infty$, A= -ve

This mean, there are two sign changes (from +ve to –ve and from –ve to +ve) . In other words two poles in the right hand side of s-plane, therefore the system is unstable.

Prob.4. The open loop transfer function of a unity negative feedback control system shown below, find the number of poles in the left half ,right half of s-plane and on imaginary axis(jw).

$$G_{open\,loop}(s) = \frac{128}{S(S^7 + 3S^6 + 10S^5 + 24S^4 + 48S^3 + 96S^2 + 128S + 192}$$

Solution:

The characteristic equation of system is

$$Q(S) = S^8 + 3S^7 + 10S^6 + 24S^5 + 48S^4 + 96S^3 + 128S^2 + 192S + 128$$

Routh's table can constructed as follows

S^8	1	10	48	128	128
S^7	3	24	96	196	0
S^6	2	16	64	128	
S^5	0	0	0	0	
S^4	?	?	?	?	
S^3	?				
S^2	?				
S^1	?				
S^0	?				

This is the second special case ,when all the row elements are zeros.

To solve this, return to first even polynomial (S^6) and form a new polynomial which is called auxiliary equation as follows:

$$P(s)=2S^6+16S^4+64S^2+128$$

But the auxiliary equation in the simplest form and this can be done for each row of the Routh's table.

$$P(s)=S^6+8S^4+32S^2+64$$

Next step ,differentiate this polynomial with respect to S to form the coefficients that replace the row of zeros:

$$\frac{dP(s)}{ds}=6S^5+32S^3+64S=0$$

Now the coefficients of S^5 in the main table will be as follows:

S^5 6 32 64

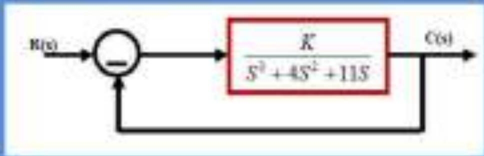
Then complete the table as in the previous examples. If your calculation is correct you find two sign changes from the even polynomial(sixth order). Hence ,the system has two right half plane poles. Because of the symmetry about the origin ,the even polynomial must have an equal number in the left half plane poles. The remaining two will be on J-w axis. There are no sign change from the beginning of the table down to the even polynomial(sixth order). Therefore the rest of the polynomial has no right half plane poles.

The final result will be two poles in the right half, four poles in the left half and two poles on the imaginary axis. Hence the system is unstable.

In the Matlab ,we will come to the closed loop control system and the code will be as follows:

```
numg=128;
deng =[1 3 10 24 . . .
48 96 128 196 0]
G1=tf(numg,deng);
G=feedback(G1,1)
poles=pole(G)
```

Prob.5. For the system shown below, find the minimum possible values of K at which the system is unstable?



Solution:

C.E= 1+O/L.T.F

Routh's array

S^3	1	11	0
S^2	4	K	0
S^1	$(44-K)/4$	0	
S^0	K		

O/L.T.F= G(s).H(s); H(s)=1;

C.E. = 1+ K/s^3+4s^2 +11s= s^3+4s^2 +11s+K=0

For stable sys. (44 – K)/ 4 $\geq$ 0

For unstable sys. (44 – K)/ 4 $\leq$ 0

Therefore; the min. value for stable is

K $\geq$ 44.

H.W.1. Consider the following characteristic equation:

$$s^4 + 2s^3 + (4 + K)s^2 + 9s + 25 = 0$$

Using the Routh stability criterion, determine the range of K for stability? ((Ans: K > 6.056))

H.w.2. Consider the closed-loop system shown in Figure 1, If K/J=4, what is the value of K_h will yield the damping ratio to be 0.6? (($K_h = 0.6$))

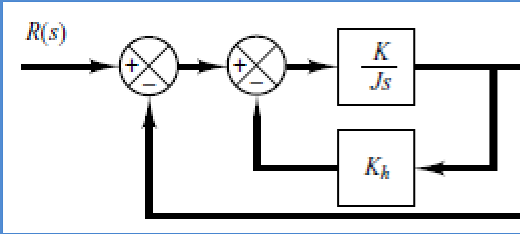


Figure 1

H.w.3. Consider the closed-loop system shown in Figure 2. Determine the range of K for stability. Assume that K>0.

((Ans: 12.5>k>0))

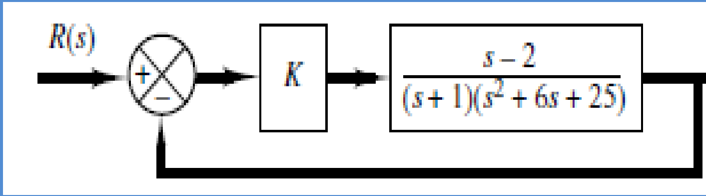


Figure 2

University of Anbar
College of Engineering
Dept. of Electrical Engineering

Lecture No. Eight

Root Locus.

8.1. Introduction.

8.2. General Rules of Root Locus.

8.3. Examples.

Root Locus Inventor

Walter Richard Evans (January 15, 1920 – July 10, 1999) was a noted American control theorist and the inventor of the root locus method in 1948. He was the recipient of the 1987 American Society of Mechanical Engineers Rufus Oldenburger Medal and the 1988 AACC's Richard E. Bellman Control Heritage Award.

8.1. Introduction.

To facilitate the application of the root-locus method, the following rules are established for $K > 0$. These rules are based upon the interpretation of the angle condition and an analysis of the characteristic equation. These rules can be extended for the case where $K < 0$. The rules for both $K > 0$ and $K < 0$ are listed in Sec. 7.16 for easy reference. The rules presented aid in obtaining the root locus by expediting the plotting of the locus. The root locus can also be obtained by using the MATLAB program. These rules provide checkpoints to ensure that the computer solution is correct. They also permit rapid sketching of the root locus, which provides a qualitative idea of achievable closed-loop system performance.

8.2. General Rules of Root Locus.

Rule 1: Number of Branches of the Locus:

The characteristic equation C.E.(s)=1+G(s)H(s)=0 is of degree n=mu; therefore, there are n roots. As the open-loop sensitivity K is varied from zero to infinity, each root traces a continuous curve. Since there are n roots, there are the same numbers of curves or branches in the complete root locus. Since the degree of the polynomial C.E.(s) is determined by the poles of the open-loop transfer function, the number of branches of the root locus is equal to the number of poles of the open-loop transfer function.

Rule 2: Real-Axis Locus:

In Fig. 1 are shown a number of open-loop poles and zeros. If the angle condition is applied to any search point such as s1 on the real axis, the angular contribution of all the poles and zeros on the real axis to the left of this point is zero. The angular contribution of the complex-conjugate poles to this point is 360°. (This is also true for complex-conjugate zeros.) Finally, the poles and zeros on the real axis to the right of this point each contribute 180° (with the appropriate sign included). From Eq.(1) the angle of G(s)H(s) to the point s1 is given by

$$\beta = \sum(\text{angles of denominator terms}) - \sum(\text{angles of numerator terms})$$
$$= \begin{cases} (1+2h)180^\circ & \text{for } K > 0 \\ h360^\circ & \text{for } K < 0 \end{cases} \quad \text{-- (1)}$$

or

$$\phi_0 + \phi_1 + \phi_2 + \phi_3 + [(\phi_4)_{+j} + (\phi_4)_{-j}] - (\psi_1 + \psi_2) = (1+2h)180^\circ$$
$$180^\circ + 0^\circ + 0^\circ + 0^\circ + 360^\circ - 0^\circ - 0^\circ = (1+2h)180^\circ$$

Therefore, s1 is a point on a branch of the locus. Similarly, it can be shown that the point s2 is not a point on the locus. The poles and zeros to the left of a point s on the real axis and the 360° contributed by the complex-conjugate poles or zeros do not affect the odd-multiple-of-180° requirement. Thus, if the total number of real poles and zeros to the right of a search point s on the real axis is odd, this point lies on the locus. In

Fig.1 the root locus exists on the real axis from p0 to p1, z1 to p2, and p3 to z2. All points on the real axis between z1 and p2 in Fig.1 satisfy the angle condition and are therefore points on the root locus.

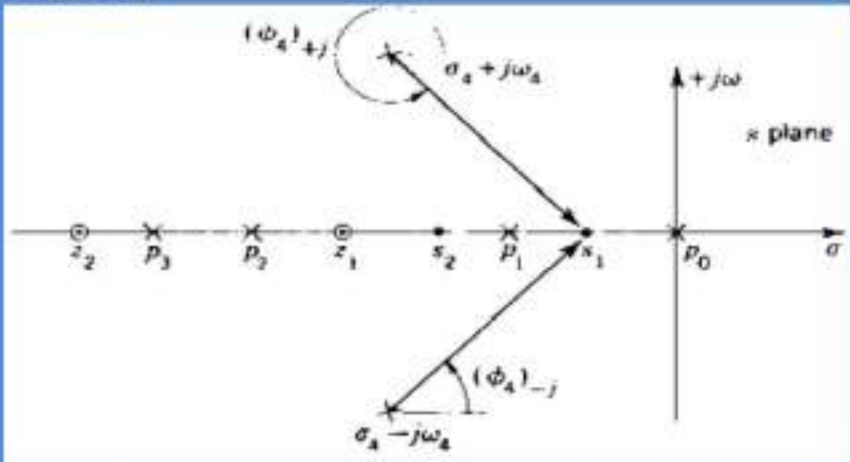


Fig.1. Determination of the real-axis locus.

However, there is no guarantee that this section of the real axis is part of just one branch. Fig.2 a illustrate the situation where part of the real axis between a pole and a zero is divided into three sections that are parts of three different branches.

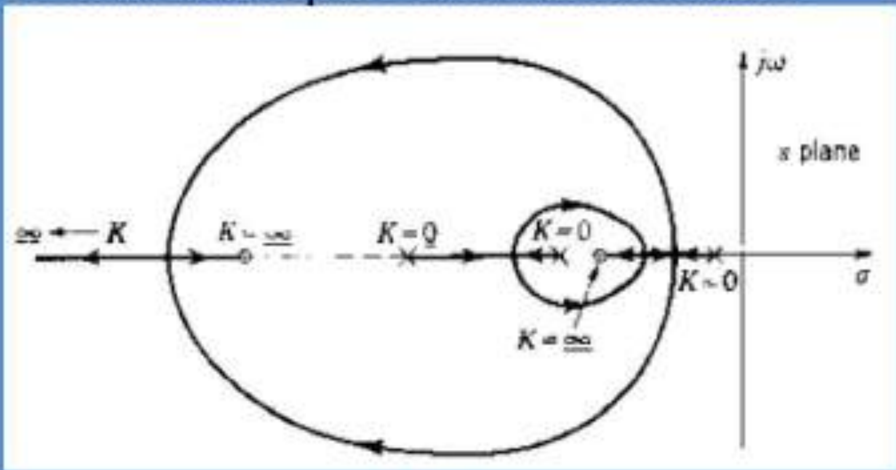


Fig.2.

Rule 3: Locus End Points:

The magnitude of the loop sensitivity that satisfies the magnitude condition is given by Eq.(3) and has the general form in Eq.(4),

$$|K| = \frac{|s^m| \cdot |s-p_1| \cdot |s-p_2| \cdots |s-p_u|}{|s-z_1| \cdots |s-z_w|} = \text{loop sensitivity} \quad \text{--- (3)}$$

$$|W(s)| = K = \frac{\prod_{c=1}^{n} |s-p_c|}{\prod_{h=1}^{w} |s-z_h|} \quad \text{----- (4)}$$

Since the numerator and denominator factors of Eq.(4) locate the poles and zeros, respectively, of the open-loop transfer function, the following conclusions can be drawn:

1) When s=pc (the open-loop poles), the loop sensitivity K is zero.
2) When s=zh (the open-loop zeros), the loop sensitivity K is infinite.

When the numerator of Eq.(4) is of higher order than the denominator, then s=1 also makes K infinite, thus being equivalent in effect to a zero. Thus, the locus starting points (K=0) are at the open-loop poles and the locus ending points (K=1) are at the open-loop zeros (the point at infinity being considered as an equivalent zero of multiplicity equal to the quantity n - w).

Rule 4: Asymptotes of Locus as s Approaches Infinity

Plotting of the locus is greatly facilitated by evaluating the asymptotes approached by the various branches as s takes on large values. Taking the limit of G(s)H(s) as s approaches infinity, based on Eqs.(5) and (6), yields

$$G(s)H(s) = \frac{K(s-z_1)\cdots(s-z_w)}{s^m(s-p_1)\cdots(s-p_u)} = \frac{K\prod_{h=1}^{w}(s-z_h)}{s^m\prod_{c=1}^{u}(s-p_c)} \qquad \text{------ (5)}$$

$$G(s)H(s) = \frac{K(s-z_1)\cdots(s-z_w)}{s^m(s-p_1)\cdots(s-p_u)} = -1 \qquad \text{---------(6)}$$

$$\lim_{s\to\infty} G(s)H(s) = \lim_{s\to\infty}\left[K\frac{\prod_{h=1}^{n}(s-z_h)}{\prod_{c=1}^{n}(s-p_c)}\right] = \lim_{s\to\infty}\frac{K}{s^{n-w}} = -1$$ ------(7)

Remember that K in Eq.(7) is still a variable in the manner prescribed previously, thus allowing the magnitude condition to be met. Therefore, as s →∞, There are n-w asymptotes of the root locus, and their angles are given by

$$-K = s^{n-w}$$

$$|-K| = |s^{n-w}|$$ Magnitude condition

$$\angle -K = \angle s^{n-w} = (1+2h)180^\circ$$ Angle condition ----(8)

Rewriting Eq. (9) gives $(n-w)\angle s = (1+2h)180^\circ$ or

$$\gamma = \frac{(1+2h)180^\circ}{n-w} \quad \text{as } s \to \infty$$ --------(9)

$$\gamma = \frac{(1+2h)180^\circ}{[\text{number of poles of } G(s)H(s)] - [\text{number of zeros of } G(s)H(s)]}$$
---(10)

Eq.(10) reveals that, no matter what magnitude s may have, after a sufficiently large value has been reached, the argument (angle) of s on the root locus remains constant. For a search point that has a sufficiently large magnitude, the open-loop poles and zeros appear to it as if they had collapsed into a single point. Therefore, the branches are asymptotic to straight lines whose slopes and directions are given by Eq. (10) (see Fig.3). These asymptotes usually do not go through the origin. The correct real-axis intercept of the asymptotes is obtained from Rule 5.

$s_1 = |s_1| \angle 1$

$s_2 = |s_2| \angle 2$

.

.

$s_x = |s_x| \angle x$

.

$s_x = \infty \angle \gamma$

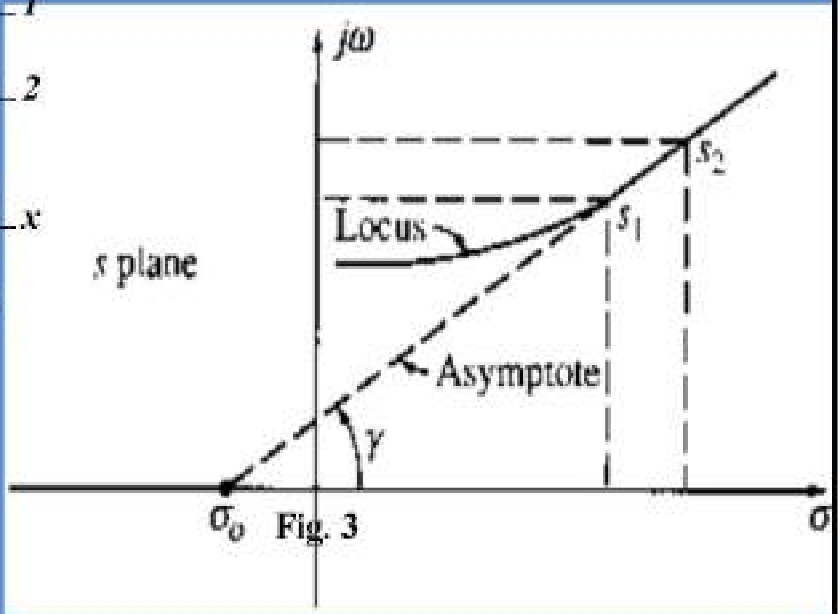


Fig. 3

Rule 5: Real-Axis Intercept of the Asymptotes

The real-axis crossing so of the asymptotes can be obtained by applying the theory of equations. The result is

$$\sigma_o = \frac{\sum_{c=1}^{n} \mathrm{Re}(p_c) - \sum_{h=1}^{w} \mathrm{Re}(z_h)}{n - w}$$

The asymptotes are not dividing lines, and a locus may cross its asymptote. It may be valuable to know from which side the root locus approaches its asymptote. The locus lies exactly along the asymptote if the pole-zero pattern is symmetric about the asymptote line extended through the point so. Rule 6: Breakaway Point

Rule 6: Breakaway Point on the Real Axis

The branches of the root locus start at the open-loop poles where K=0 and

end at the finite open-loop zeros or at s=1.When the root locus has branches

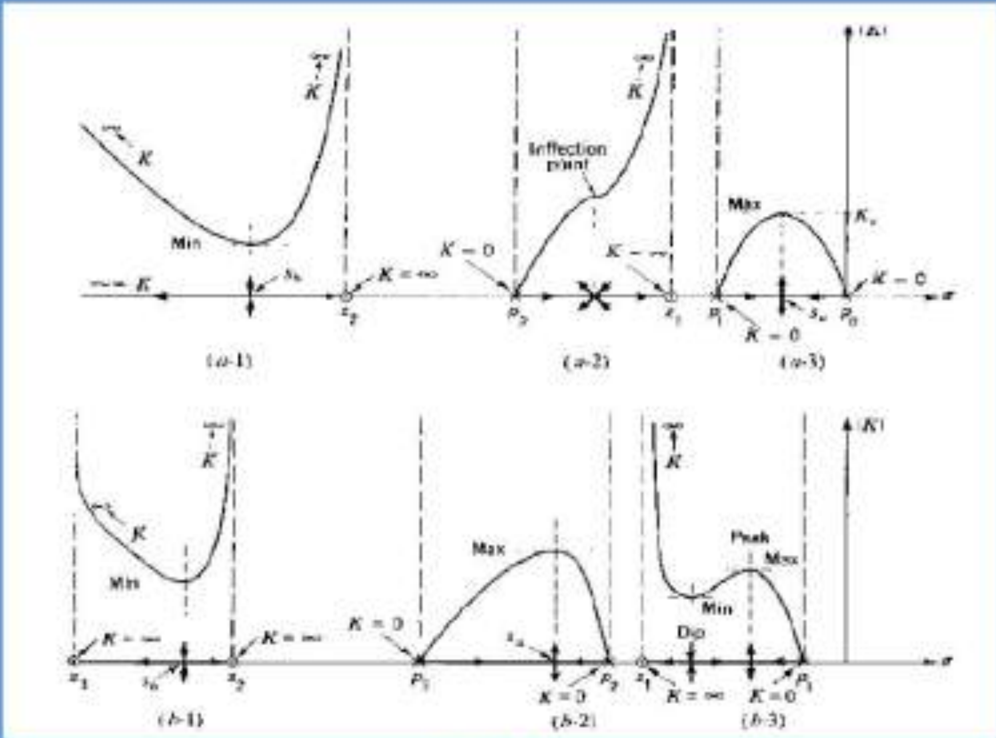


Fig.4.

on the real axis between two poles, there must be a point at which the two branches breakaway from the real axis and enter the complex region of the s plane in order to approach zeros or the point at infinity. (Examples are shown in Fig. 4.a-3: between p0 and p1, and in Fig. 4.b-2: between p2 and p3.) For two finite zeros (see Fig. 4.b-1) or one finite zero and one at infinity (see Fig. 4.a-1) the branches are coming from the complex region and enter the real axis. In Fig. 4.a-3 between two poles there is a point s a for which the loop sensitivity K z is greater than for points on either side of s a on the real axis.

In other words, since K starts with a value of zero at the poles and increases in value as the locus moves away from the poles, there is a point somewhere in between where the K's for the two branches simultaneously reach a maximum value. This point is called the breakaway point. Plots of K vs. s utilizing Eq.(3) are shown in Fig.4 for the portions of the root locus that exist on the real axis for K > 0. The point sb for which the value of K is a minimum between two zeros is called the break-in point. The breakaway and break-in points can easily be calculated for an open-loop pole-zero combination for which the derivatives of W(s)=K is of the second order. As an example, if

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

then

$$W(s) = s(s+1)(s+2) = -K \qquad \text{------------- (11)}$$

Multiplying the factors together gives

$$W(s) = s^3 + 3s^2 + 2s = -K$$

When s^3+s^2+2s is a minimum, -K is a minimum and K is a maximum. Thus, by taking the derivative of this function and setting it equal to zero, the points can be determined:

$$\frac{dW(s)}{ds} = 3s^2 + 6s + 2 = 0$$

or

$$s_{a,b} = -1 \pm 0.5743 = -0.4257,\ -1.5743$$

Since the breakaway point's s a for K > 0 must lie between s=0 and s=-1, in order to satisfy the angle condition, the value is s_a =-0:4257;

The other point, sb=-1.5743, is the break-in point on the root locus forK <0.

Substituting s_a= 0.4257 into Eq. (11) gives the value of K at the breakaway

Point for K > 0 as

$$K = -[(-0.426)^3 + (3)(-0.426)^2 + (2)(-0.426)] = 0.385$$

When the derivative of W(s) is of higher order than 2,a digital-computer program can be used to calculate the roots of the numerator polynomial of dW(s)/ds; these roots locate the breakaway and break-in points. Note that it is possible to have both a breakaway and a break-in point between a pole and zero (finite or infinite) on the real axis, as shown in Figs.4a-1, 7.11a-2, and 4.b-3. The plot of Kvs. s for a locus between a pole and zero falls into one of the following categories:

1. The plot clearly indicates a peak and a dip, as illustrated between p1 and z1 in Fig. 4.b-3. The peak represents a 'maximum' value of K that identifies a break-in point.
2. The plot contains an inflection point. This occurs when the breakaway and break-in points coincide, as is the case between p2 and z1 in Fig. 4a-2.

3. The plot does not indicate a dip-and-peak combination or an inflection point. For this situation there are no break-in or breakaway points.

The next geometrical shortcut is the rapid determination of the direction in which the locus leaves a complex pole or enters a complex zero. Although in Fig.5.a complex pole is considered, the results also hold for a complex zero.

In Fig.5.a, an area about p2 is chosen so that l2 is very much smaller than l0, l1, l3, and (l)1. For illustrative purposes, this area has been enlarged many times in Fig.5b. Under these conditions the angular contributions from all the other poles and zeros, except p2, to a search point anywhere in this area are approximately constant. They can be considered to have values determined as if the search point were right at p2. Applying the angle condition to this small area yields

$$\phi_0 + \phi_1 + \phi_2 + \phi_3 - \psi_1 = (1 + 2h)180^\circ$$

or the departure angle is

$$\phi_{2_D} = (1 + 2h)180^\circ - (\phi_0 + \phi_1 + 90^\circ - \psi_1)$$

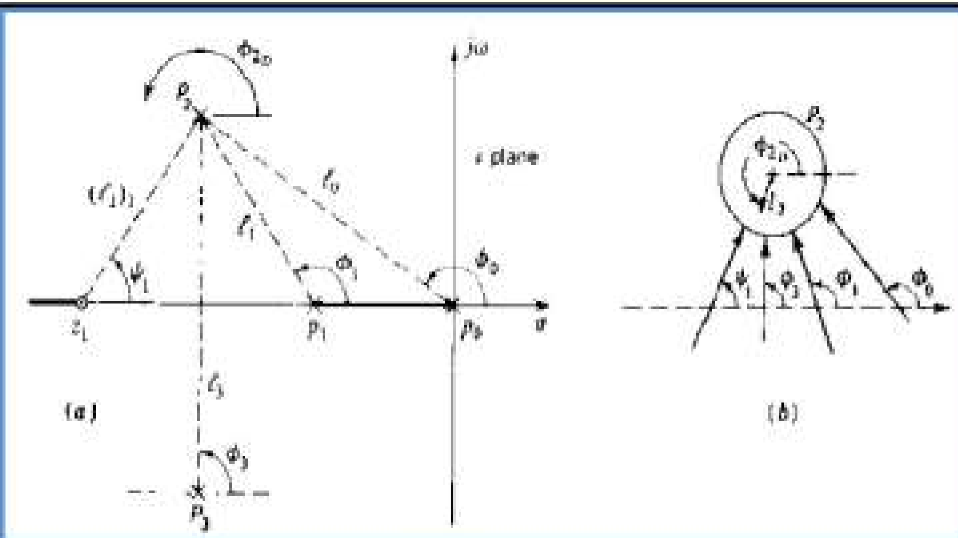


Fig. 5

In a similar manner the approach angle to a complex zero can be determined. For an open-loop transfer function having the pole-zero arrangement shown in Fig.6, the approach angle c1 to the zero z1 is given by

$$\psi_{1_A} = (\phi_0 + \phi_1 + \phi_2 - 90^{\circ}) - (1 + 2h)180^{\circ}$$

In other words, the direction of the locus as it leaves a pole or approaches a zero can be determined by adding up, according to the angle condition, all the
angles of all vectors from all the other poles and zeros to the pole or zero in question. Subtracting this sum from $(1+2h)180^{\circ}$ gives the required direction.

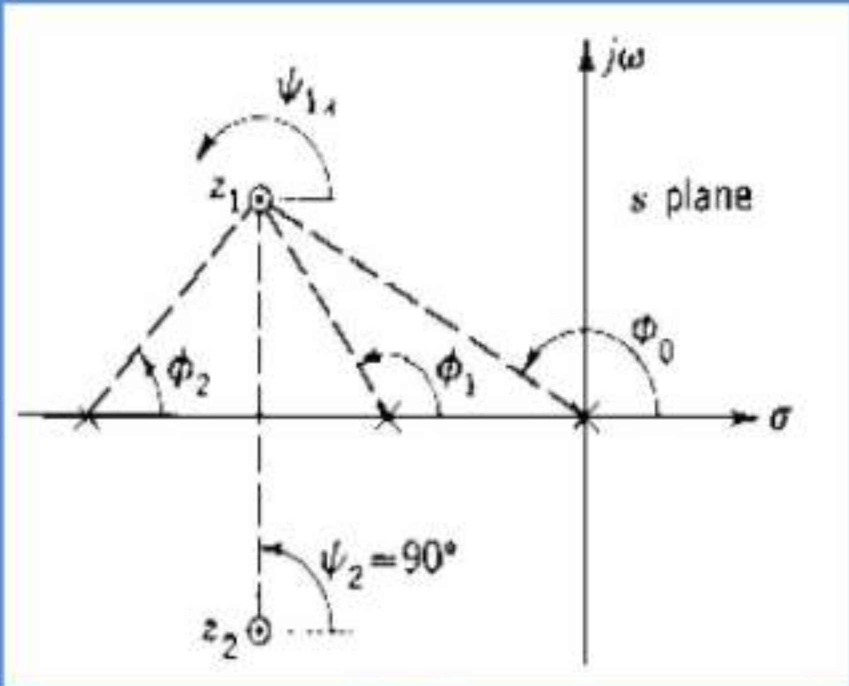


Fig. 6

Rule 7: Complex Pole (or Zero): Angle of Departure:

The next geometrical shortcut is the rapid determination of the direction in which the locus leaves a complex pole or enters a complex zero. Although inFig.7.a complex pole is considered, the results also hold for a complex zero.

In Fig. 7.a, an area about p2 is chosen so that l2 is very much smaller than l0, l1, l3, and (l) 1. For illustrative purposes, this area has been enlarged many times in Fig. 7.b. Under these conditions the angular contributions from all the other poles and zeros, except p2, to a search point anywhere in this area are approximately constant. They can be considered to have values determined as if the search point were right at p2.Applying the angle condition to this small area yields. In a similar manner the approach angle to a complex zero can be determined. For an open-loop transfer function having the pole-zero

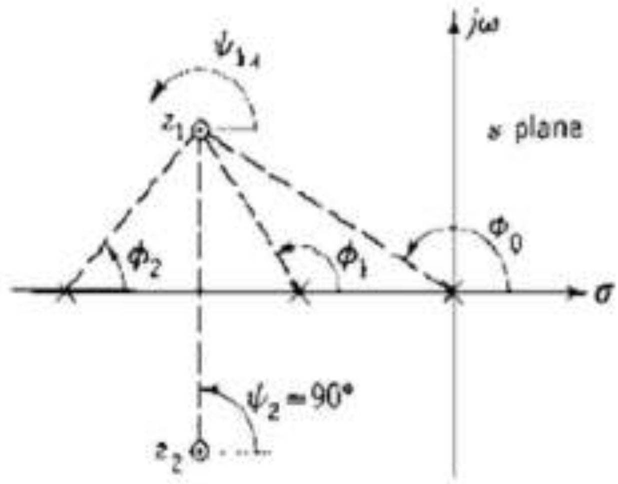


Fig. 8. Angle condition in the vicinity of a complex zero.

Arrangement shown in Fig.8, the approach angle c1 to the zero z1 is given by

$$\psi_{1_s} = (\phi_0 + \phi_1 + \phi_2 - 90^{\circ}) - (1 + 2h)180^{\circ}$$

In other words, the direction of the locus as it leaves a pole or approaches a zero can be determined by adding up, according to the angle condition, all the angles of all vectors from all the other poles and zeros to the pole or zero in question. Subtracting this sum from (1+2h)180 gives the required direction.

Rule 8: Imaginary-Axis Crossing Point:

In cases where the locus crosses the imaginary axis into the right-half s plane,

the crossover point can usually be determined by Routh's method or by similar means. For example, if the closed-loop characteristic equation D1D2+ N1N2=0 is of the form.

$$s^3 + bs^2 + cs + Kd = 0$$

the Routhian array is

s^3	1	c
s^2	b	Kd
s^1	$(bc - Kd)/b$	
s^0	Kd	

An undamped oscillation may exist if the s_1 row in the array equals zero. For this condition the auxiliary equation obtained from the s 2 row is

$$bs^2 + Kd = 0$$

and its roots are

$$s_{1,2} = \pm j\sqrt{\frac{Kd}{b}} = \pm j\omega_n \qquad \text{-----(12)}$$

The loop sensitivity term K is determined by setting the s1 row to zero:

$$K = bc/d$$

For K > 0, Eq. (12) gives the natural frequency of the undamped oscillation. This corresponds to the point on the imaginary axis where the locus crosses over into the right-half s plane. The imaginary axis divides the s plane into stable and unstable regions. Also, the value of K from Eq. ($K = bc/d$) determines the value of the loop sensitivity at the crossover point. For values of K < 0 the term in the s0 row is negative, thus characterizing an unstable response. The limiting values for a stable response are therefore

$$0 < K < bc/d$$

In like manner, the crossover point can be determined for higher-order characteristic equations. For these higher-order systems care must be exercised in analyzing all terms in the first column that contain the term K in order to obtain the correct range of values of gain for stability.

Rule 9: Intersection or Non-intersection of Root-Locus Branches:

The theory of complex variables yields the following properties:

1. A value of s that satisfies the angle condition of Eq. (1) is a point on the root locus. If $dW(s)/ds \neq 0$ at this point, there is one and only one branch of the root locus through the point.
2. If the first y_1 derivatives of W(s) vanish at a given point on the root locus, there are y branches approaching and y branches leaving this point; thus, there are root-locus intersections at this point. The angle between two adjacent approaching branches is given by

$$\lambda_y = \pm \frac{360^\circ}{y}$$

Also, the angle between a branch leaving and an adjacent branch that is approaching the same point is given by

$$\theta_y = \pm \frac{180^\circ}{y}$$

Fig. 9. illustrates these angles at s=-3 ,with θy=45° and λy=90°.

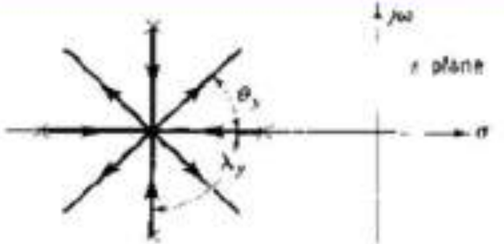


Fig. 9. Root locus for

$$G(s)H(s) = \frac{K}{(s+2)(s+4)(s^2+6s+10)}.$$

Ex.(1).

Find $C(s)/R(s)$ with $\zeta = 0.5$ for the dominant roots (roots closest to the imaginary axis) for the feedback control system represented by

$$G(s) = \frac{K_1}{s(s^2/2600 + s/26 + 1)} \quad \text{and} \quad H(s) = \frac{1}{0.04s + 1}$$

Rearranging gives

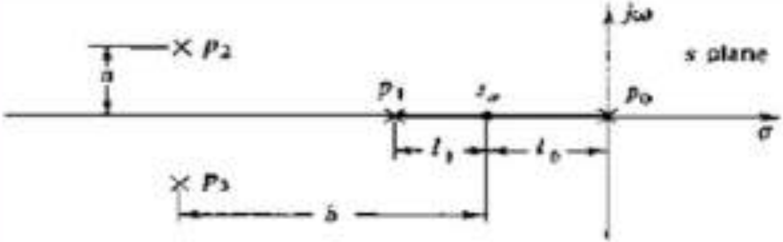
$$G(s) = \frac{2600K_1}{s(s^2 + 100s + 2600)} = \frac{N_1}{D_1} \quad \text{and} \quad H(s) = \frac{25}{s + 25} = \frac{N_2}{D_2}$$

Thus,

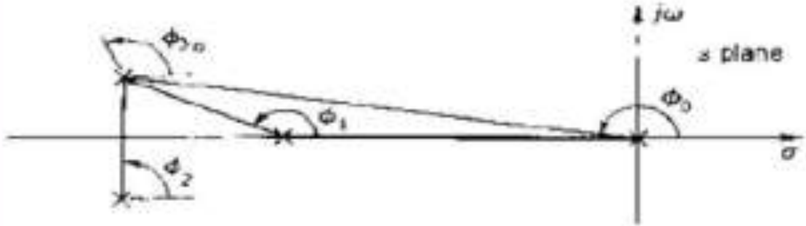
$$G(s)H(s) = \frac{65{,}000K_1}{s(s+25)(s^2 + 100s + 2600)} = \frac{K}{s^4 + 125s^3 + 5100s^2 + 65{,}000s}$$

where $K = 65{,}000K_1$.

1. The poles of $G(s)H(s)$ are plotted on the s plane in Fig. below the values of these poles are $s = 0, -25, -50 + j10, -50 - j10$.



Location of the breakaway point.



Determination of the departure angle.

The system is completely unstable for $K < 0$. Therefore, this example is solved only for the condition $K > 0$.

2. There are four branches of the root locus.
3. The locus exists on the real axis between 0 and -25.
4. The angles of the asymptotes are

$$\gamma = \frac{(1+2h)180^\circ}{4} = \pm 45^\circ, \pm 135^\circ$$

5. The real-axis intercept of the asymptotes is

$$\sigma_o = \frac{0 - 25 - 50 - 50}{4} = -31.25$$

6. The breakaway point s_a on the real axis between 0 and -25 is found by solving $dW(s)/ds = 0$

$$-K = s^4 + 125s^3 + 5100s^2 + 65{,}000s$$

$$\frac{d(-K)}{ds} = 4s^3 + 375s^2 + 10{,}200s + 65{,}000 = 0$$

$$s_a = -9.15$$

7. The angle of departure ϕ_{3_D} from the pole $-50 + j10$ is obtained from

$$\phi_0 + \phi_1 + \phi_2 + \phi_{3_D} = (1+2h)180^\circ$$

$$168.7^\circ + 158.2^\circ + 90^\circ + \phi_{3_D} = (1+2h)180^\circ$$

$$\phi_{3_D} = 123.1^\circ$$

Similarly, the angle of departure from the pole $-50 + j10$ is -123.1°.

8. The imaginary-axis intercepts are obtained from

$$\frac{C(s)}{R(s)} = \frac{2600K_1(s+25)}{s^4 + 125s^3 + 5100s^2 + 65{,}000s + 65{,}000K_1}$$

The Routhian array for the denominator of $C(s)/R(s)$, which is the characteristic polynomial, is

$$\begin{array}{c|lll} s^4 & 1 & 5100 & 65{,}000K_1 \\ s^3 & 1 & 520 \text{ (after division by 125)} & \\ s^2 & 1 & 14.2K_1 \text{ (after division by 4580)} & \\ s^1 & 520 - 14.2K_1 & & \\ s^0 & 14.2K_1 & & \end{array}$$

Pure imaginary roots exist when the s^1 row is zero. This occurs when $K_1 = 520/14.2 = 36.6$. The auxiliary equation is formed from the s^2 row:

$$s^2 + 14.2K_1 = 0$$

and the imaginary roots are

$$s = \pm j\sqrt{14.2K_1} = \pm j\sqrt{520} = \pm j\,22.8$$

9. Additional points on the root locus are found by locating points that satisfy the angle condition

$$\angle s + \angle s+25 + \angle s+50-j10 + \angle s+50+j10 = (1+2m)180^\circ$$

The root locus is shown in Fig.10

10. The radial line for $\zeta = 0.5$ is drawn on the graph of Fig. 10 at the angle

$$\eta = \cos^{-1} 0.5 = 60^\circ$$

The dominant roots obtained from the graph are

$$s_{1,2} = -6.6 \pm j11.4$$

11. The gain is obtained from the expression

$$K = 65{,}000K_1 = |s| \cdot |s+25| \cdot |s+50-j10| \cdot |s+50+j10|$$

Inserting the value $s_1 = -6.6 + j11.4$ into this equation yields

$$K = 65{,}000K_1 = 598{,}800$$

$$K_1 = 9.25$$

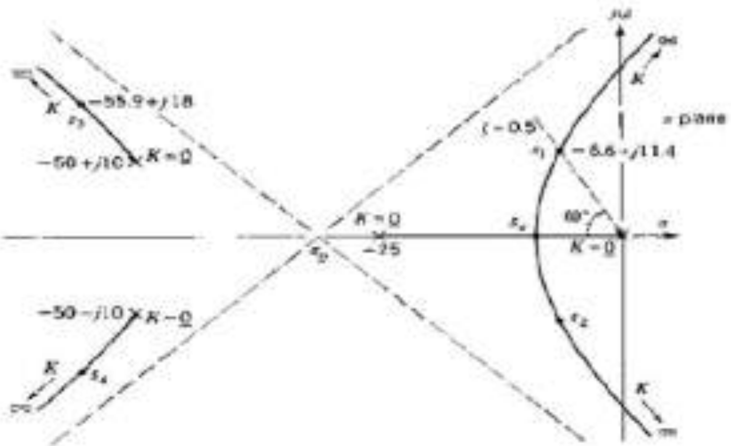


Fig 10. Root locus for $G(s)H(s) = \dfrac{65{,}000K_1}{s(s+25)(s^2+100s+2600)}$

12. The other roots are evaluated to satisfy the magnitude condition $K = 598{,}800$. The remaining roots of the characteristic equation are

$$s_{3,4} = -55.9 \pm j18.0$$

The real part of the additional roots can also be determined by using the rule from Eq. (7.68):

$$0 - 25 + (-50 + j10) + (-50 - j10)$$
$$= (-6.6 + j11.4) + (-6.6 - j11.4) + (\sigma + j\omega_d) + (\sigma - j\omega_d)$$

This gives $\sigma = -55.9$

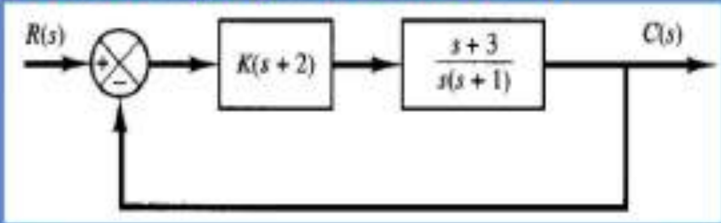
By using this value, the roots can be determined from the root locus as $-55.9 \pm j18.0$.

13. The control ratio, using values of the roots obtained in steps 10 and 12, is

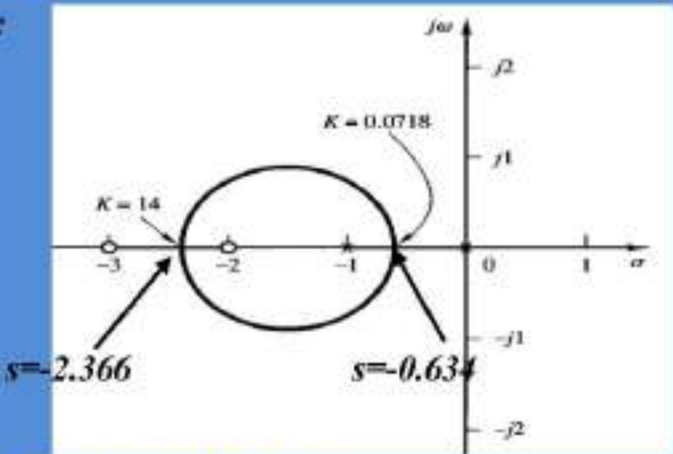
$$\frac{C(s)}{R(s)} = \frac{N_1 D_2}{\text{factors determined from root locus}}$$

$$= \frac{1}{(s + 6.6 + j11.4)(s + 6.6 - j11.4)}$$
$$= \frac{24{,}040(s + 25)}{(s + 55.9 + j18)(s + 55.9 - j18)}$$
$$= \frac{24{,}040(s + 25)}{(s^2 + 13.2s + 173.5)(s^2 + 111.8s + 3450)}$$

Ex.(2). Plot Root Loci for system shown below:



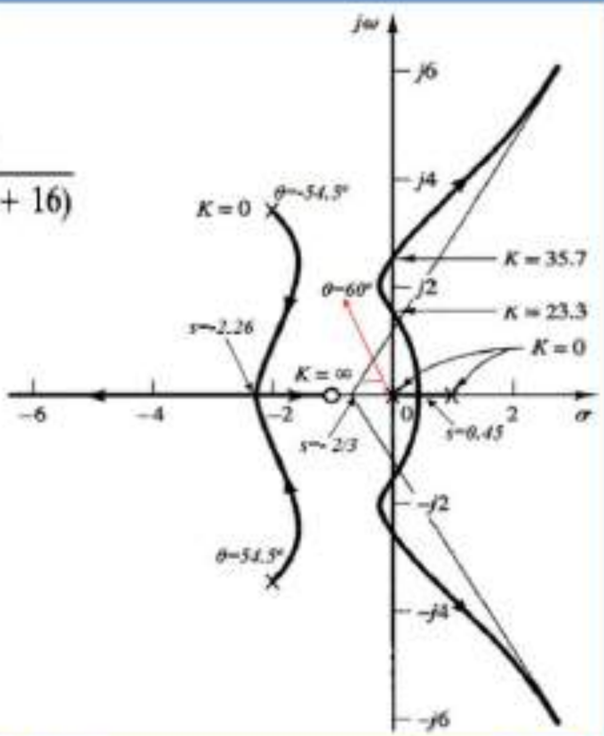
Solution:



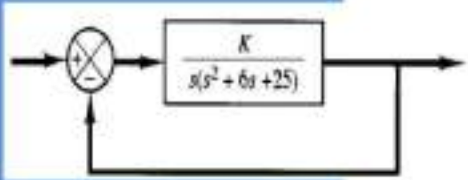
Ex.(3). Plot the root loci for the system has the following T.F:

$$G(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}$$

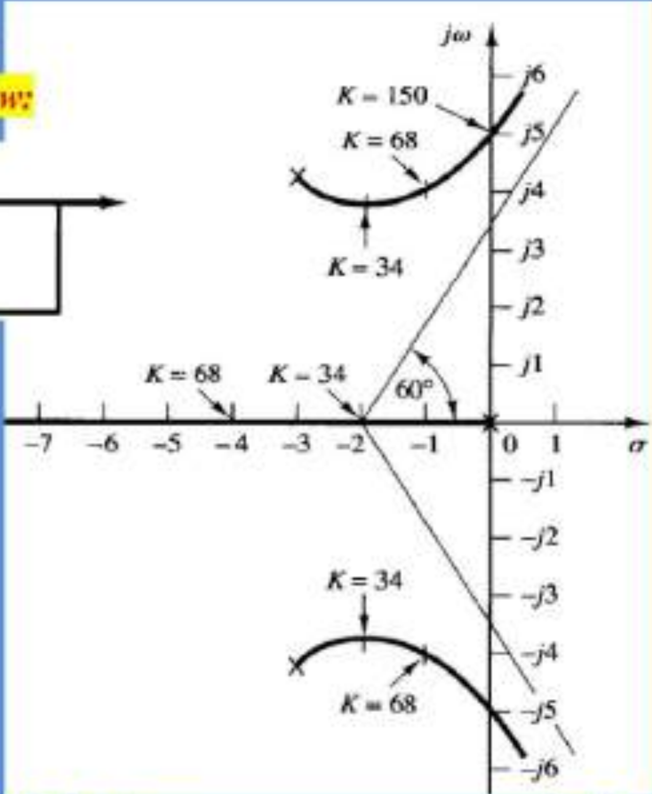
Solution:



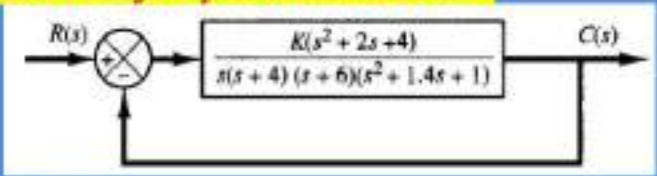
Ex.(4). Plot Root Loci for system shown below:



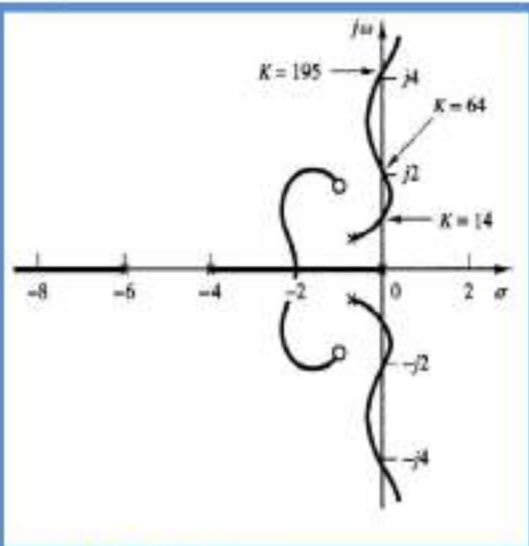
Solution:



Ex.(5). Plot Root Loci for system shown below:



Solution:



Ex.(6). Plot the root loci for the system has the following T.F:

$$G(s)\,H(s) = \frac{K(s^2 - 1)}{\left(s^2 + 1\right)\left(s^2 + 4\right)}$$

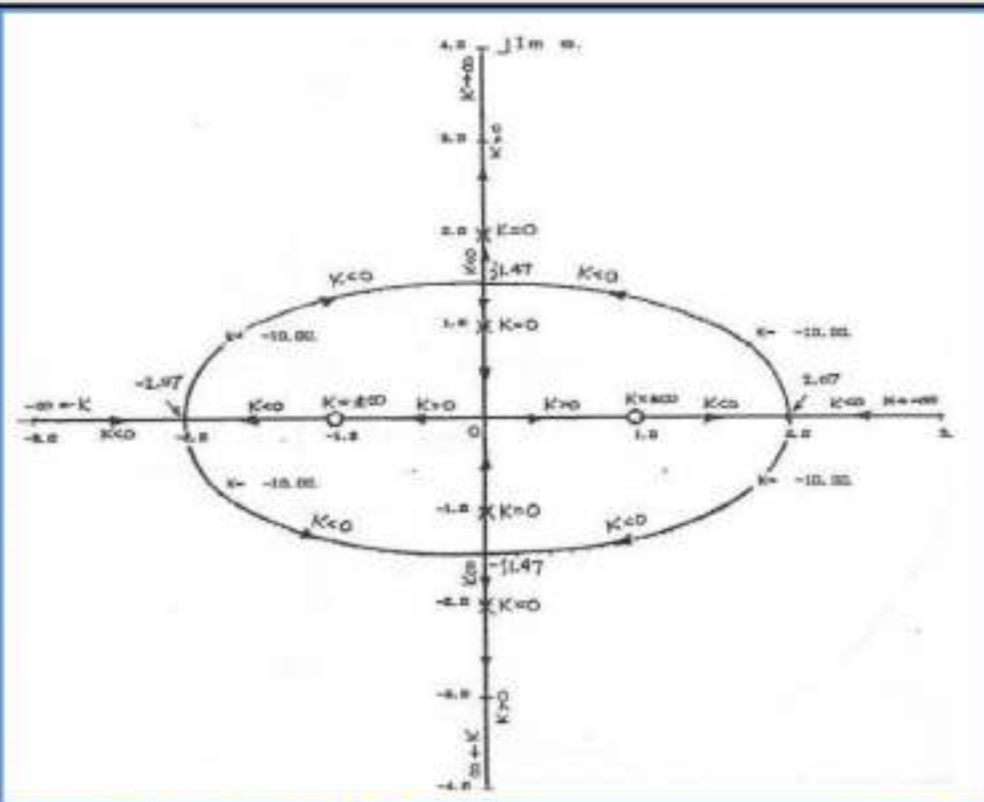
Solution:

Asymptotes: $K > 0$: 90^{0}, 270^{0} $K < 0$: 0^{0}, 180^{0}

Intersect of Asymptotes: $\sigma_1 = \frac{-1+1}{4-2} = 0$

Breakaway-point Equation: $s^5 - 2s^3 - 9s = 0$

Breakaway Points: -2.07, 2.07, $-j1.47$, $j1.47$


jIm
K=0
j1.47
-j1.47
K<0
K>0
2.07
0

Ex.(7). Plot the root loci for the system has the following T.F:

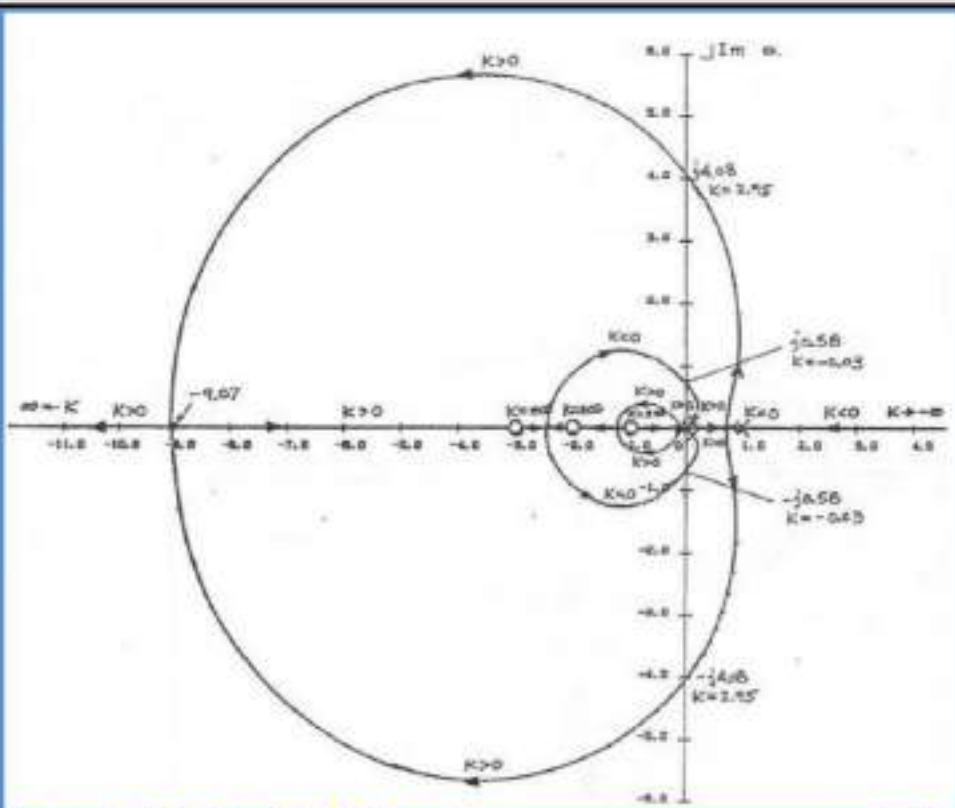
$$G(s)H(s) = \frac{K(s+1)(s+2)(s+3)}{s^3(s-1)}$$

Solution:

Asymptotes: $K > 0$: 180^0 $K < 0$: 0^0

Breakaway-point Equation: $s^6 + 12s^5 + 27s^4 + 2s^3 - 18s^2 = 0$

Breakaway Points: $-1.21,\ -2.4,\ -9.07,\ 0.683,\ 0,\ 0$


jIm
K>0
-9.07
K<0
j0.58
-j0.58
j4.08
K=2.95
-j4.08
K=2.95